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This report contains documentation for four computer programs used in the imperfection sensitivity analysis of cylindrical shells. The four programs are based upon Donnell's equation for cylindrical shells. The formulation of each program is worked in detail so others may make modification. Input and output instructions, together with the program listing, are provided. The report is in two volumes. Volume I discusses the development and applications of the programs, and Volume II is the user's manual.

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NUMERICAL METHODS FOR IMPERFECTION SENSITIVITY ANALYSIS OF STIFFENED CYLINDRICAL SHELLS, VOL. I - DEVELOPMENT AND APPLICATIONS

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Volume I

NUMERICAL METHODS FOR IMPERFECTION SENSITIVITY ANALYSIS OF STIFFENED CYLINDRICAL SHELLS.

VOL. 1: Development and Applications

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September 1986

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This report has been reviewed by the Office of Public Affairs (ASD/PA) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

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1.0 INTRODUCTION

The buckling analysis of shells has in recent years gained considerable interest. The current demand for lightweight efficient structures requires the engineer to closely examine the response of the shell, particularly for those states of stress that lead to buckling. It has been observed that as the design of a shell structure is optimized, the structure becomes more prone to instability. Further, small perturbations in load or geometry will not permit the structure to sustain its load, and failure will occur. In some cases, linear methods did not give satisfactory results and nonlinear theories were employed in an attempt to improve the correlation between prediction and experiment. This led to postbuckling and imperfection studies, which in turn provided a measure of the load-carrying capacity of the structure. Unfortunately, these analyses did not give an accurate prediction of the residual strength of the shell, but did give an adequate assessment of its behavior just prior to buckling.

The thin stiffened and unstiffened circular cylindrical shell has been extensively used in the aerospace and energy industries. The cylinder was found to be particularly imperfection-sensitive to axial compression. This sensitivity was attributed to the deviation of the shell's geometry from a "perfect" shell. This deviation from the true or perfect shape is termed an imperfection. Thus, the phrase "imperfection sensitivity" has come into popular use.

During the initial phase of this research, a review of the literature was conducted and a report prepared [1]. In this review, the analysis methods derived by Donnell and Koiter were closely examined. Donnell derived a set of nonlinear equations that govern the response of an axially loaded cylinder. Koiter examined the behavior of the shell at adjacent states of stress to the buckling load and hypothesized the buckling characteristics through these states. Koiter's theory is based on the

initial postbuckling behavior of a perfect structure. The theory leads to an equation for a load versus postbuckling displacements:

$$\lambda/\lambda^* = 1 + a\varepsilon + b\varepsilon^2 \tag{1}$$

in which λ/λ^* is the ratio of applied load to the classical buckling load, ε is the amplitude of displacement of the buckled mode, and a and b are coefficients that depend on the geometry of the structure, prebuckling behavior, and type of loading. Equation (1) applies to cases with unique eigenvalues.

The coefficient b is more important than the coefficient a in most cases involving thin shells; therefore, the coefficient a usually takes on a value of zero, particularly for shells of revolution. Often referred to as Koiter's parameter, the coefficient b essentially determines the imperfection sensitivity characteristics of the shell. When b is less than zero, λ/λ^* is less than unity, thus predicting a reduction in load-carrying capacity from that of the perfect shell. Computer methods are often required for the evaluation of the imperfection sensitivity parameter b. Only one computer program system, written by Cohen [2], is specifically designed to examine the imperfection sensitivity of general shells of revolution. In using this program, one first has to execute a static program module to determine the prestress state. These results are passed to a stability program which uses a subroutine to establish the imperfection sensitivity of the shell. Although Cohen's procedure appears to be correct, it has not been widely accepted by the engineering community because the program is not user-friendly. There remains a need for an efficient computer procedure that is easy to use and yet suitable for preliminary design.

In the following sections the theoretical basis of four such computer programs are presented. The programs are designed to evaluate the buckling characteristics of a cylindrical shell stiffened by rings or by rings and stringers. Equivalent orthotropic membrane and bending stiffnesses are derived through a

model in which the ring and stringer stiffeners are "smeared" over the shell. For light and medium stiffening, where the cross-section area of the stiffener divided by the stiffener spacing is less than one-half the thickness of the shell, the smeared model has been found to be adequate.

For ease of identification, each computer procedure has the project name "PVRC" followed by the initial of the principal contributor:

PVRCB	Boros [3]
PVRCH	Hutchinson [4]
PVRCK	Khot [5]
PVRCA	Arbocz [6]

The capabilities of these programs are listed in Table 1. Because of the applicable theories employed, only PVRCH considers the boundary conditions at the ends of the cylinder. For the remaining procedures, simply-supported boundary conditions are approximated.

All four programs are based on Donnell's equations, but different solution procedures are employed, and different loading, imperfection geometry, and boundary conditions allowed. Each of these procedures will be discussed in detail.

The first program, PVRCB, examines the buckling characteristics of a constant thickness circular cylinder stiffened by rings only. The imperfection is assumed to be axisymmetric and the effects of the boundary condition are not considered. A secondary option is included for a two-mode imperfection (one axisymmetric and the other nonaxisymmetric). The applied load can only be an axial stress.

The second program, PVRCH, examines the buckling strength of a circular cylinder stiffened by stringers and rings. The shell can have either clamped or simply supported conditions. The loading can be either an axial stress or normal pressure. Only an axisymmetric imperfection geometry can be considered.

The third program, PVRCK, also considers a ring and stringer stiffened circular cylinder; however, the cylinder is comprised

TABLE 1. IMPERFECTION SENSITIVITY PROGRAMS

Program	Capability Parameters					
	Rings	Stringers	Boundary Conditions	Layer (no./type)	Loading	Imperfection Geometry
PVRCB	x			1/ISO	AX	symmetric nonsymmetric
PVRCH	x	x	SS, CL	1/ISO	AX,P	symmetric
PVRCK	x	x	~ SS	5/ORTHO	AX,P,T	nonsymmetric
PVRCA	x	x	~ SS	1/ISO	AX	arbitrary

Legend

AX	axial load
P	pressure
T	torsion
ISO	isotropic
ORTHO	orthotropic
SS	simply supported
CL	clamped

of a layered orthotropic material. Boundary conditions are not imposed. The loading can be any combination of axial compression, pressure and torsion. An imperfection amplitude acting in each critical longitudinal and circumferential response mode is assumed and the buckling load is computed.

The fourth program, PVRCA, uses Koiter's method for calculating the buckling load of axially compressed stiffened cylindrical shells with a given asymmetric imperfection. Boundary conditions are ignored. No other loads are considered.

Fortran listings of the four computer programs are included as appendixes (see Volume II of this report). In addition, common driver subroutines are listed in Appendix E. These programs are designed to execute in an interactive mode. Therefore, no preparation of an input data file is required. The user is automatically prompted. Copies of the CRT displays are provided to illustrate the use of each program.

The governing equations of a cylindrical shell of revolution are presented in Section 2.1. The four sections that subsequently follow are the detailed derivations that form the basis of the solution procedures for PVRCB, PVRCH, PVRCK, and PVRCA, respectively. The significance of certain assumptions as to loading, boundary conditions, construction of the shell's wall, etc. will also be discussed at the appropriate point in the development.

Numerical examples obtained from the first three developed computer programs for four ring stiffened and four stringer stiffened configurations are presented in Section 3.0. A separate verification example is also presented for the program PVRCA. Section 4.0 discusses the behavior of the imperfection sensitivity parameter and the applicability and usefulness of the four programs. Finally, conclusions and some observations are presented.

2.0 TECHNICAL DISCUSSION

2.1 Governing Equations

Consider a cylindrical shell of radius R, length L, and thickness, t. If the usual assumption of thinness is made ($(t/R)^2 \ll 1$), then the governing equations of thin shell theory can be developed. For the coordinate system shown in Figure 1, approximate nonlinear strain-displacement and curvature-displacement equations can be written as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{matrix} U_{,x} + \frac{1}{2}(W_{,x})^2 \\ V_{,y} + \frac{1}{2}(W_{,y})^2 - \frac{W}{R} \\ U_{,y} + V_{,x} + W_{,x} W_{,y} \end{matrix} \qquad (2)$$

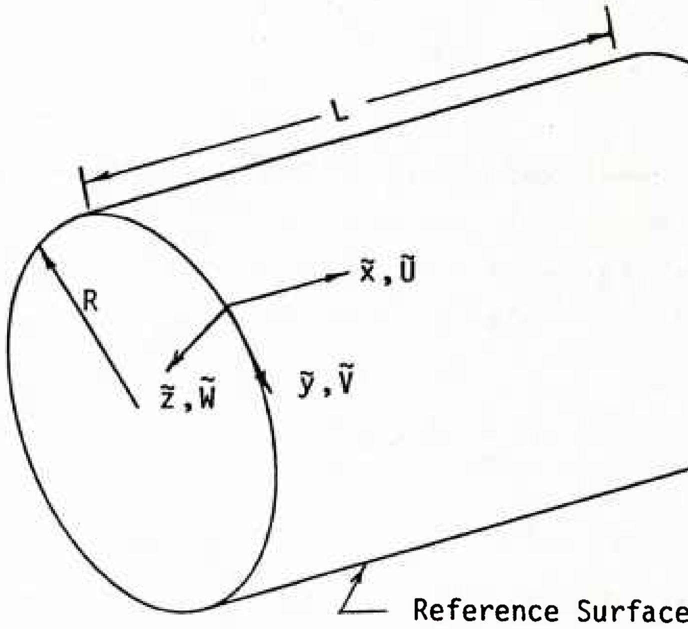
$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{matrix} W_{,xx} \\ W_{,yy} \\ 2\,W_{,xy} \end{matrix}$$

in which U, V, W are the axial, circumferential and radial displacements of the reference surface, and the comma denotes differentiation with respect to the subscripted variable.

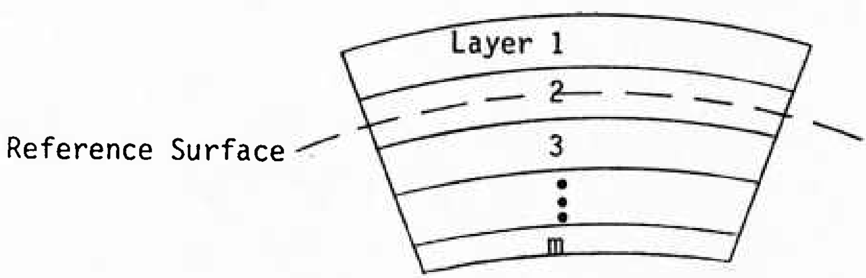
The equilibrium equations derived by Donnell are:

$$\begin{aligned} &N_{x,x} + N_{xy,y} = 0 \\ &N_{y,y} + N_{xy,x} = 0 \\ &\frac{1}{R} N_y + N_x W_{,xx} + 2\,N_{xy} W_{,xy} + N_y W_{,yy} \\ &\qquad + M_{x,xx} + 2\,M_{xy,xy} + M_{y,yy} = 0 \end{aligned} \qquad (3)$$

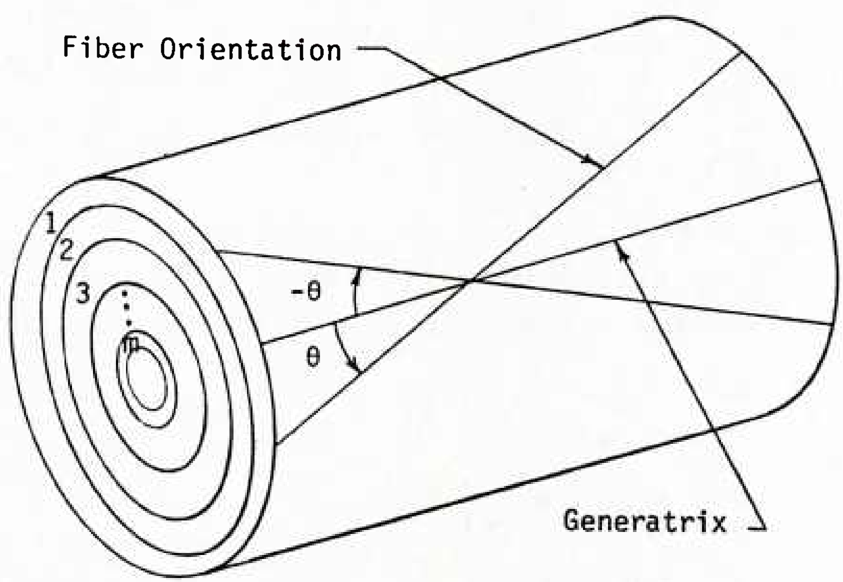
where $N = \{N_x, N_y, N_{xy}\}^T$ (the stress resultants) and $M = \{M_x, M_y, M_{xy}\}^T$ (the moment resultants). These resultants are usually defined along the reference surface of the shell and are



(a)



(b)



(c)

Figure 1 Geometry of the Shell

the average and linear variation of the inplane stresses through the thickness of the shell, respectively.

The constitutive equations for a layered orthotropic shell of revolution can be written in matrix form as:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} C & K \\ K^T & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \tag{4}$$

where the membrane, coupling, and bending stiffness of the composite shell can be defined, respectively, as:

$$C_{ij} = \sum_{k=1}^{N} B_{ij}^{(k)} (h_{k+1} - h_k)$$

$$K_{ij} = \frac{1}{2} \sum_{k=1}^{N} B_{ij}^{(k)} (h_{k+1}^2 - h_k^2)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} B_{ij}^{(k)} (h_{k+1}^3 - h_k^3)$$

$B_{ij}^{(k)}$ = compliance moduli in each layer k

(e.g., $B_{11} = \frac{E_{11}}{1-\nu_1 \nu_2}$)

h_k = distance from reference surface to each layer k

The matrix coefficients C_{ij}, K_{ij}, and D_{ij} can be modified to include the effects of the stiffening element:

$$\begin{aligned} \hat{C}_{ij} &= C_{ij} + \delta_{ij} \frac{EA}{b} \\ \hat{K}_{ij} &= K_{ij} + \delta_{ij} \frac{EAe}{b} \\ \hat{D}_{ij} &= D_{ij} + \delta_{ij} \frac{EI}{b} \end{aligned} \tag{5}$$

where

δ_{ij} = Kronecker delta function
E = modulus of elasticity of the stiffener
A = cross-sectional area of the stiffener
b = distance between stiffeners
e = eccentricity of the stiffener
I = moment of inertia of the stiffener

If i=j=1, the above properties are for stringers; if i=j=2, they represent ring properties and both are "smeared" over the span b. When $i \neq j$ the Poisson's effect of a shell should be considered. Flügge [7] develops an anisotropic approximation for a stiffened shell but does not include Poisson's effect. Hutchinson and Amazigo [8], however, developed equations which account for a partial effect.

In order to examine the imperfection sensitivity of a shell, the imperfection geometry must be introduced into Eqs. (2) and (3). The radial variation from a perfect form is usually taken as the imperfection, $\hat{W}$. The total radial displacement in Eqs. (2) and (3) is therefore comprised of a response and an imperfection. Thus, additional terms must be included in the above relationships so that there is no stress when at zero load.

2.2 Program PVRCB

This program examines the buckling characteristics of a constant thickness isotropic circular cylinder. Only ring stiffeners are considered, and their properties are smeared as described in the previous section. The out-of-plane bending stiffness and the torsional rigidity of the ring are neglected. The applied load can only be an axial stress (σ), although an applied pressure (p) will be carried during the derivation. Boundary conditions are not explicitly stated, but simply supported conditions are implied.

The development of the governing equations and the basic solution procedure is based upon the works of Tennyson, his students [3], and colleagues [9]. Boros [3] presents the derivation for the effects of the axisymmetric imperfection, and Hutchinson [9] presents the effects of a nonaxisymmetric imperfection.

Assume an Airy's stress function F, defined as:

$$
\begin{aligned}
N_x &= F_{,yy} \\
N_y &= F_{,xx} \\
N_{xy} &= -F_{,xy}
\end{aligned}
\tag{6}
$$

The Donnell-von Karman shell equations [10] take the form, for compatibility:

$$
\begin{aligned}
L_H[F] - L_Q[W] - W^2_{,xy} + W_{,xx} W_{,yy} + \hat{W}_{o,xx} W_{,yy} + \hat{W}_{o,yy} W_{,xx} \\
- 2 \hat{W}_{o,xy} W_{,xy} = 0
\end{aligned}
\tag{7}
$$

and for equilibrium:

$$
\begin{aligned}
L_D[W] + L_Q[F] - F_{,xx} W_{,yy} - F_{,yy} W_{,xx} + 2 F_{,xy} W_{,xy} \\
- F_{,xx} \hat{W}_{o,yy} - F_{,yy} \hat{W}_{o,xx} + 2 F_{,xy} \hat{W}_{o,xy} + p = 0
\end{aligned}
\tag{8}
$$

where

$$L_H[\;] = H_{xx}[\;]_{,xxxx} + 2\,H_{xy}[\;]_{,xxyy} + H_{yy}[\;]_{,yyyy}$$

$$L_Q[\;] = Q_{xx}[\;]_{,xxxx} + 2Q_{xy}[\;]_{,xxyy} + Q_{yy}[\;]_{,yyyy} + \frac{1}{R}[\;]_{,xx}$$

$$L_D[\;] = D_{xx}[\;]_{,xxxx} + 2D_{xy}[\;]_{,xxyy} + D_{yy}[\;]_{,yyyy} \qquad (9)$$

The formulas for effective stretching and bending stiffnesses are given in Ref. [8] and repeated herein in Table 2.

2.2.1 Axisymmetric Imperfection

Assume an axisymmetric imperfection of the form:

$$\hat{W}_o = -\,\xi\,t\cos\frac{2p_o x}{R} \qquad (10)$$

where

$$p_o = (2\,k + 1)\,\frac{\pi R}{2L} \qquad k = 1,2,3\;...$$

L = length of the shell

ξ_t = imperfection amplitude

t = thickness of the shell

Prior to buckling, the radial deflection and stress function are assumed to be of the form:

$$W = \frac{\nu}{E}\,\sigma R + W^*$$
$$F = -\,\frac{1}{2}\,\sigma t y^2 + F^* \qquad (11)$$

where

σ = axial stress

$$W^* = B\cos\frac{2p_o x}{R}$$
$$F^* = A\cos\frac{2p_o x}{R} \qquad (12)$$

TABLE 2

DEFINITION OF STRESS COEFFICIENTS USED BY HUTCHINSON

$$(D_{xx}, D_{xy}, D_{yy}) = \frac{Et^3}{12(1-\nu^2)} - (\bar{D}_{xx}, \bar{D}_{xy}, \bar{D}_{yy}) \; ; \; (Q_{xx}, Q_{xy}, Q_{yy}) = t(\bar{Q}_{xx}, \bar{Q}_{xy}, \bar{Q}_{yy})$$

$$(H_{xx}, H_{xy}, H_{yy}) = \frac{1}{Et}(\bar{H}_{xx}, \bar{H}_{xy}, \bar{H}_{yy}) \; ; \; (B_{xx}, B_{xy}, B_{yx}, B_{yy}) = Et^2(\bar{B}_{xx}, \bar{B}_{xy}, \bar{B}_{yx}, \bar{B}_{yy})$$

$\bar{B}_{xx} = \alpha_s\gamma_s(1+\alpha_r)/\alpha_o$	$\bar{B}_{yy} = \alpha_r\gamma_r(1+\alpha_s)/\alpha_o$	$\bar{B}_{xy} = \nu\alpha_s\alpha_r\gamma_r/\alpha_o$	$\bar{B}_{yx} = \nu\alpha_r\alpha_s\gamma_s/\alpha_o$
$\bar{D}_{xx} = 1+\beta_s +$ $+ [12(1-\nu^2)\alpha_s(1+\alpha_r)\gamma_s^2]/\alpha_o$	$\bar{D}_{yy} = 1+\beta_r +$ $+ [12(1-\nu^2)\alpha_r(1+\alpha_s)\gamma_r^2]/\alpha_o$	$\bar{D}_{xy} = 1 +$ $+ [12(1-\nu^2)\nu\alpha_s\alpha_r\gamma_s\gamma_r]/\alpha_o$	$\bar{D}_{yx} = \bar{D}_{xy}$
$\bar{H}_{xx} = [1+\alpha_s(1-\nu^2)]\alpha_o$	$\bar{H}_{yy} = [1+\alpha_r(1-\nu^2)]\alpha_o$	$\bar{H}_{xy} = (1+\nu)-\nu/\alpha_o$	$\bar{H}_{yx} = \bar{H}_{xy}$
$\bar{Q}_{xx} = \nu\alpha_s\gamma_s/\alpha_o$	$\bar{Q}_{yy} = \nu\alpha_r\gamma_r/\alpha_o$	$\bar{Q}_{xy} = -[\alpha_s\gamma_s(1+(1-\nu^2)\alpha_r) +$ $+ \alpha_r\gamma_r(1+(1-\nu^2)\alpha_s)]/2\alpha_o$	$\bar{Q}_{yx} + \bar{Q}_{xy}$

$$\alpha_s = A_s/d_s t \; , \; \beta_s = EI_s/Dd_s \; , \; \gamma_s = e_s/t$$

$$\alpha_r = A_r/d_r t \; , \; \beta_r = EI_r/Dd_r \; , \; \gamma_r = e_r/t$$

$$\alpha_o = (1+\alpha_s)(1+\alpha_r) - \nu^2\alpha_s\alpha_r$$

Substituting Eq. (10) and (11) into (7) and (8), leads to:

$$\frac{Et^3}{12(1-\nu^2)} W^*_{,xxxx} + \frac{1}{R} F^*_{,xx} + \frac{4p_o^2 \xi \sigma t^2}{R^2} \cos \frac{2p_o x}{R} + \sigma t W^*_{,xx} = 0$$

$$\frac{1}{Et(1+\alpha_r)} F^*_{,xxxx} - \frac{1}{R} W^*_{,xx} = 0 \tag{13}$$

Substituting Eq. (12) into (13) yields the coefficients A and B:

$$A = \frac{12(1-\nu^2)\xi\sigma t^3 R^2 E(1+\alpha_r)}{C} \tag{14}$$

$$B = \frac{-48(1-\nu^2)\xi\sigma t^2 p_o^2 R^2}{C} \tag{15}$$

where

$$C = 4E[4p_o^4 t^3 + 3(1+\alpha_r)R^2(1-\nu^2)t - \frac{3\sigma t(1-\nu^2)\ p_o^2 R^2}{E}]$$

The deformation just prior to buckling is governed by Eq. (11). When buckling starts, the modal deformation pattern becomes nonaxisymmetric. The modal pattern may change as the postbuckling process continues. The initial postbuckling state is assumed to be of the form:

$$W = \frac{\nu\sigma R}{E} + B \cos \frac{2p_o x}{R} + w \tag{16}$$

$$F = -\frac{\sigma t y^2}{2} + A \cos \frac{2p_o x}{R} + \frac{Et^3}{c} f \tag{17}$$

where

$$c = \sqrt{3(1-\nu^2)}$$

and w and f are infinitesimal buckling modal quantities representing the displacement and stress functions. Substituting Eqs. (16) and (17) into (7) and (8) yields the following compatibility (18) and equilibrium (19) equations for the infinitesimal buckling modal quantities w and f:

$$\frac{t^2}{c(1+\alpha_r)} f_{,xxxx} + \frac{2t^2(1+\alpha_r(1+\nu))}{4(1+\alpha_r)} f_{,xxyy}$$

$$+ \frac{t^2(1+\alpha_r(1-\nu^2))}{c(1+\alpha_r)} f_{,yyyy} + \frac{t\alpha_r\gamma_r}{(1+\alpha_r)} w_{,xxyy}$$

$$- \frac{\nu t\alpha_r\ \gamma_r}{1+\alpha_r} w_{,yyyy} - \frac{1}{R} w_{,xx} + \frac{4p_o^2}{R^2}(\xi t+\Gamma) \tag{18}$$

$$\cdot\ \cos\left(\frac{2p_o x}{R}\right) w_{,yy} = 0$$

$$\frac{Et^3}{12(1-\nu^2)} w_{,xxxx} + \frac{2Et^3}{12(1-\nu^2)} w_{,xxyy} + \frac{Et^3}{12(1-\nu^2)}[1+\beta_r$$

$$+ \frac{12(1-\nu^2)\alpha_r\gamma_r}{1+\alpha_r}] w_{,yyyy} - \left(\frac{\alpha_r\gamma_r}{1+\alpha_r}\right)\frac{Et^4}{c} f_{,xxyy}$$

$$+ \nu\left(\frac{\alpha_r\gamma_r}{1+\alpha_r}\right)\frac{Et^4}{c} f_{,yyyy} + \frac{1}{R}\frac{Et^3}{c} f_{,xx}$$

$$- \frac{Et^3}{12(1-\nu^2)}\frac{16p_o^4}{R^4}\Gamma\cos\frac{2p_o x}{R} - \frac{Et(1+\alpha_r)}{R^2}\Gamma\cos\frac{2p_o x}{R}$$

$$+ \frac{4\ p_o^2\xi\alpha t^2}{R^2}\cos\frac{2p_o x}{R} - \frac{4p_o^2 E}{R^2}\frac{\xi t^4}{c}\cos\frac{2p_o x}{R} f_{,yy} \tag{19}$$

$$+ \frac{4p_o^2\Gamma\sigma t}{R^2}\cos\frac{2p_o x}{R} + \sigma t w_{,xx}$$

$$- \frac{4p_o^2 \Gamma E t^3}{cR^2} \cos \frac{2p_o x}{R} f_{,yy}$$

$$+ (1+\alpha_r) \, Et\Gamma \cos \frac{2p_o x}{R} w_{,yy} = 0$$

where

$$\Gamma = \frac{48(1-\nu^2)\xi\sigma t^2 R^2 p_o^2}{C}$$

Koiter [11] argues that for the unstiffened shell the nonsymmetric buckling pattern can have the form:

$$w = t \sum_{m=1}^{N} C_m \cos [(2m-1)p_o \, x/R] \cos ny/R \tag{20}$$

He suggests that the prebuckling stress function F (Eq. 11) is periodic and therefore the prebuckling stresses are periodic, which implies a periodic distribution of circumferential tensile, or circumferential compressive membrane stresses. "Buckling in a nonsymmetric mode will be stimulated in regions of compressive circumferential stresses and hampered in regions of tensile stresses. We may therefore expect the existence of an asymmetric buckling mode with nodal lines where the circumferential stresses attain their maximum." The axial period of such a mode is twice the period of the axisymmetric prebuckling state response. Thus, at

$$x = \pm \frac{L}{2} = (2\kappa+1) \pi R/2 p_o \quad , \quad \text{where } \kappa \text{ is an integer}$$

simply supported edges are assumed. This condition is incompatible with Eq. (10), but Koiter suggests that it is of no great consequence, provided that

$$\frac{p_o^2 t}{2Rc} > 0.04$$

Substituting Eq. (20) into (18) results in a fourth order partial differential equation in f with constant coefficients. The solution is of the form:

$$f = \sum_{m=0}^{\infty} \left[\frac{A_1 C_m + A_2(C_{m-1} + C_{m+1})}{A_3}\right] \cos \frac{(2m-1)p_o x}{R} \cos \frac{ny}{R} \qquad (21)$$

where

$$A_1 = \frac{\alpha_r \gamma_r t^2 (2m-1) p_o^2 n^2}{(1+\alpha_r)\, R^4} + \frac{t^2 \nu \alpha_r \gamma_r}{(1+\alpha_r)} \frac{n^4}{R^4} - \frac{t(2m-1)^2 p_o^2}{R^3}$$

$$A_2 = \frac{2p_o^2 n^2}{R^4} [\xi t+B]$$

$$A_3 = \frac{t^2}{c(1+\alpha_r)} \frac{(2m-1)^4 p_o^4}{R^4} + \frac{2(1+\alpha_r(1+\nu))t^2}{c(1+\alpha_r)} \frac{(2m-1)^2 p_o^2 n^2}{R^4}$$

$$+ \frac{1+\alpha_r(1-\nu^2)}{1+\alpha_r} \frac{t^2}{c} \frac{n^4}{R^4}$$

Since Eq. (20) cannot exactly satisfy either the compatibility equations or the equilibrium equations, we assume f can be made to satisfy the compatibility equation provided that the coefficients of Eq. (20) are determined in some mathematical least square sense. This is accomplished by substituting Eq. (21) and (20) into (19), resulting in an error function ε:

$$\varepsilon(x,y) = \varepsilon(A_i, C_m, \cos \frac{(2m-1)p_o x}{R} \cos \frac{2p_o x}{R} \;, \cos \frac{ny}{R}) \qquad (22)$$

First, the trigometric function in the axial direction can be condensed into a single function. Second, a Galerkin procedure will be applied as follows:

$$\int_0^{2\pi R}\int_{-\ell/2}^{\ell/2} \varepsilon(x,y)\cos\frac{(2q-1)p_o x}{R}\cos\frac{ny}{R}\,dxdy = 0 \qquad (23)$$

$$q = 1,2,3\ldots$$

Thus the infinite set is formed:

$$[\frac{Et^4}{12(1-\nu^2)}\sum C_q \frac{(2q-1)^4 p_o^4}{R^4} + \frac{2Et^4}{12(1-\nu^2)}\frac{n^2}{R^2}\sum C_q \frac{(2q-1)^2 p_o^2}{R^2}$$

$$+ \frac{Et^4}{12(1-\nu^2)}\frac{n^4}{R^4}(1+\beta_r + \frac{12(1-\nu^2)\alpha_r\gamma_r}{1+\alpha_r})\sum C_q$$

$$- \frac{Et^4}{(1+\alpha_r)c}\frac{n^2}{R^2}\sum \frac{A_1C_q + A_2(C_{q-1}+C_{q+1})}{A_3}\frac{(2q-1)^2p_o^2}{R^2}$$

$$+ \frac{\nu Et\alpha_r\gamma_r}{(1+\alpha_r)c}\frac{n^4}{R^4}\sum\frac{A_1C_q + A_2(C_{q-1}+C_{q+1})}{A_3}$$

$$- \frac{Et^3}{Rc}\sum\frac{A_1C_q + A_2(C_{q-1}+C_{q+1})}{A_3}\frac{(2q-1)^2p_o^2}{R^2}$$

$$- \sigma t^2\sum C_q\frac{(2q-1)^2p_o^2}{R^2}]\ [\frac{R(\pi+2)}{2}\{\frac{R}{(q-j)2p_o}\sin\frac{(q-j)2p_oL}{R}$$

$$+ \frac{R}{(q+j-1)2p_o}\sin\frac{(q+j-1)2p_oL}{R}\}] - [\frac{4p_o^2t\Gamma n^2}{R^4}\sum C_q$$

$$- \frac{4p_o^2Et^3(\xi t+\Gamma)n^2}{R^4c}\sum\frac{A_1C_q + A_2(C_{q-1}+C_{q+1})}{A_3}]$$

$$\cdot\ \frac{R(\pi+2)}{4}\{\frac{R}{(q-j-1)2p_o}\sin\frac{(q-j-1)2p_oL}{R}$$

$$+ \frac{R}{(q+j-2)2p_o} \sin \frac{(q+j-2)2p_oL}{R}$$

$$+ \frac{R}{(q-j+1)2p_o} \sin \frac{(q-j+1)2p_oL}{R}$$

$$+ \frac{R}{(q+j)2p_o} \sin \frac{(q+j)2p_oL}{R} \} = 0 \qquad (24)$$

If one retains r equations and sets $C_q = 0$ when $q > r$, a system of r homogeneous linear equations are obtained. The determinant of this final system is set equal to zero. With the notation

$$\lambda = \frac{\sigma Rc}{Et}$$

$$K^2 = \frac{p_o^2 t}{2Rc}$$

and

$$\tau^2 = \frac{n^2 t}{Rc}$$

the characteristic equation can be written in the form:

$$H_1 \lambda^3 + H_2 \lambda^2 + H_3 \lambda + H_4 = 0 \qquad (25)$$

where

$$H_1 = 1$$

$$H_2 = - \left[\frac{Q_1^2}{8K^2} + \frac{2H_5^4}{Q_2^2 K^2 (1+\alpha_r)} + \frac{H_6}{4K^2} + \frac{c\xi\tau^2(1+\alpha_r)}{4K^2}\right]$$

$$H_3 + H_6 \left[\frac{Q_1}{32K^4} + \frac{2H_5^4}{Q_2 K^4 (1+\alpha_r)} + \frac{H_6}{64K^4} + \frac{c\xi\tau^2(1+\alpha_r)}{32K^4} + \frac{c\xi\tau^2 H_5^2}{Q_2^2 K^2}\right]$$

$$H_4 = -H_6^2 \left[\frac{Q_1^2}{512K^6} + \frac{H_5^4}{32Q_2^2K^6(1+\alpha_r)} + \frac{c\xi\tau^2H_5^2}{8Q_2^2K^4} + \frac{c^2\xi^2\tau^4(1+\alpha_r)H_7}{8K^2}\right]$$

$$H_5^4 = [-K^2\tau^2\alpha_r\gamma_r c + \frac{1}{2}\tau^4\nu\alpha_r\gamma_r c - K^2(1+\alpha_r)]^2$$

$$Q_1^2 = 4K^4 + 4K^2\tau^2 + \tau^4(1+\beta_r + \frac{4\alpha_r\gamma_r^2c^2}{1+\alpha_r})$$

$$Q_2^2 = 4K^4 + 4K^2\tau^2(1+\alpha_r(1+\nu)) + \tau^4(1+\alpha_r(1-\nu^2))$$

$$H_6 = 16K^4 + (1+\alpha_r)$$

$$H_7 = \frac{1}{Q_2^2} + \frac{1}{Q_3^2}$$

$$Q_3^2 = 32K^4 + 36K^2\tau^2(1+\alpha_r(1+\nu)) + \tau^4(1+\alpha_r)1-\nu^2)$$

From Eq. (25), three roots (the minimum being the buckling load) can be obtained. The computer program PVRCB solves Eq. (25) for the roots λ.

2.2.2 Nonsymmetric Imperfection

Th above procedure can be repeated for the case of an imperfection geometry of the form:

$$W_o = -\xi_1 t \cos 2K_1x + \xi_2 t \cos K_1x \cos K_2y \tag{26}$$

where

$$K_1 = \pi Rmq_o/2$$

$$K_2 = \pi Rnq_o/2$$

$$q_o = (R/t)^{1/2}\,[12(1-\nu^2)]^{1/4}$$

and

m and n are integer wave numbers

The first term represents an axisymmetric mode of amplitude $\xi_2 t$. Hutchinson [9,12] assumed that the buckling mode would take the form:

$$W = W_1 \cos 2K_1 x + W_2 \cos K_1 x \cos K_2 y \tag{27}$$

and derived the characteristic equation:

$$D_3 \lambda^2 - D_2 \lambda + D_1 = 0 \tag{28}$$

where

$$D_1 = (B_1 G_1 - B_4 G_5) W_2 - (B_3 G_5 + B_4 G_4) W_2^2 - B_3 G_4 W_2^3$$

$$D_2 = (B_2 G_1 + G_2 B_1 + B_3 G_3) W_2 + B_4 G_3 + G_1 B_5$$

$$D_3 = G_2 B_2 W_2 + G_2 B_5$$

$$B_1 = \frac{K_1^2 + K_2^2)^2}{4} + \frac{K_1^4}{4(K_1^2 + K_2^2)^2} - \frac{2cK_1^2 K_2^2 \xi_1}{(K_1^2 + K_2^2)^2}$$

$$B_2 = \frac{K_1^2}{2}$$

$$B_3 = - c\, K_2^2 [\frac{2K_1^4}{(K_1^2 + K_2^2)^2} + \frac{1}{4}]$$

$$B_4 = - c\, K_2^2\, \xi_2 [\frac{K_1^4}{(K_1^2 + K_2^2)^2} + \frac{1}{4}]$$

$$B_5 = \frac{K_1^2 \xi_2}{2}$$

$$G_1 = 8\,K_1^4 + \frac{1}{2}$$

$$G_2 = 4\,K_1^2$$

$$G_3 = -\,4\,K_1^2\xi_1$$

$$G_4 = -\,c\,K_2^2\left[\frac{K_1^4}{(K_1^2 + K_2^2)^2} + \frac{1}{8}\right]$$

$$G_5 = -\,c\,K_2^2\xi_2\left[\frac{K_1^4}{(K_1^2 + K_2^2)^2} + \frac{1}{4}\right]$$

Hutchinson has shown that for $K_1 = K_2 \simeq 1/2$ a minimum λ is obtained. Equation (28) is also solved for λ by program PVRCB.

2.3 Program PVRCH

This program calculates the Koiter imperfection parameter and load factor for a ring and stringer stiffened cylinder with isotropic material. The stiffener properties are assumed to be smeared as in the previous case. Out-of-plane bending stiffness of each stiffener is ignored, but the torsional rigidity is included. The applied load can be either an axial stress or external pressure. Simply supported or clamped boundary conditions can be accommodated. Only an axisymmetric imperfection is considered.

Hutchinson and Frauenthal [4] analyze the imperfection sensitivity of an axially loaded stiffened cylinder. The general outline of the procedure has been covered by Budiansky [13]. To arrive at a solution, Budiansky states that three basic conditions have to be satisfied: (1) axisymmetric prebuckling deformation with zero imperfection, (2) nonaxisymmetric bifurcation from the prebuckling state using a classic linear bifurcation, and (3) initial postbuckling behavior.

Axisymmetric Prebuckling Deformation

Equations (7) and (8) are reduced by assuming zero imperfection:

$$L_H[F] - L_Q[W] - W_{,xy}^{2} + W_{,xx}W_{,yy} = 0 \tag{29}$$

$$L_D[W] + L_Q[F] - F_{,xx}W_{,yy} - F_{,yy}W_{,xx} + 2\,F_{,xy}W_{,xy} = 0 \tag{30}$$

where the operators are defined by Eq. (9). For axially stiffened shells, the Airy's stress function is related to the membrane stress resultant within the shell (N_x) and the stringer (N_s):

$$
\begin{aligned}
N_x+N_s &= F_{,yy} \\
N_y &= F_{,xx} \\
N_{xy} &= - F_{xy} \\
N_x &= A_{xx}F_{,xx} + A_{xy}F_{,yy} + B_{xx}W_{,xx}
\end{aligned}
\tag{31}
$$

The evaluation of the imperfection sensitivity of a cylinder subjected to an axial load requires the determination of two parameters: P_c, the critical load, and b, Koiter's imperfection parameter. The expansion of the critical load is given by Eq. (1): $P/P_{c\ell} = 1 + \varepsilon a + \varepsilon^2 b$ where $P_{c\ell}$ is the classical load. The imperfection sensitivity parameter b as derived in Ref. [4] is

$$
b = \frac{F^{(2)} *(W^{(1)},W^{(1)}) + 2\ F^{(1)} *(W^{(1)},W^{(2)})}{P_c[\dot{F}_c^o *(W^{(1)},W^{(1)}) + 2\ F^{(1)} *(\dot{W}_c^o,W^{(1)})]}
\tag{32}
$$

where the notation

$$
\begin{aligned}
A*(B,C) = \int [& A_{,xx}B_{,y}C_{,y} + A_{,yy}B_{,x}C_{,x} \\
& - A_{,xy}(B_{,x}C_{,y} + B_{,y}C_{,x})]\ ds
\end{aligned}
\tag{33}
$$

It is assumed that in the neighborhood of the bifurcation point, the deformation and stress functions can be expanded:

$$
\begin{aligned}
W &= W^o + \delta W^{(1)} + \delta^2 W^{(2)} + \ldots \\
F &= F^o + \delta F^{(1)} + \delta^2 F^{(2)} + \ldots
\end{aligned}
\tag{34}
$$

where

δ = amplitude of the imperfection

The prebuckled solution is expanded about the critical load, P_c, so that

$$W^o = W_c^o + (P-P_c)\dot{W}_c^o + \frac{1}{2}(P-P_c)^2 \ddot{W}_c^o + \ldots$$
$$F^o = F_c^o + (P-P_c)\dot{F}_c^o + \frac{1}{1}(P-P_c)^2 \ddot{F}_c^o + \ldots \tag{35}$$

where $$(\dot{\ }) = \frac{\partial(\ )}{\partial P}\Bigg|_{P=P_c}$$

For the axisymmetric prebuckling state, the deformation pattern is assumed to be

$$W^o = w^o$$

and the stress function

$$F^o = -\frac{1}{2}y^2\frac{P}{2\pi R} + f^o \tag{36}$$

where P is the applied load, and w^o and f^o are the infinitesimal axisymmetric modal quantities representing the displacement and the stress function.

For the axisymmetric prebuckling state, the governing equation is obtained by substituting Eq. (36) into (30), which yields:

$$[D_{xx} + \frac{Q_{xx}^2}{H_{xx}}]w^{o^{iv}} + [\frac{2Q_{xx}}{RH_{xx}} + \frac{P}{2\pi R}]w^{o''} + \frac{w^o}{R^2H_{xx}} = \frac{\nu A_{xy}P}{2\pi EtH_{xx}R^2} \tag{37}$$

and

$$H_{xx}f^{o''} = Q_{xx}w^{o''} + \frac{w^o}{R} - \frac{\nu A_{xy}P}{2\pi EtR} \tag{38}$$

The boundary conditions that can be imposed are either simply supported or clamped. The computer program uses half the cylinder length so that at boundary 1 either clamped or simply supported boundary conditions are imposed, and at the other end symmetrical conditions are assumed.

For a one-dimensional system of equations with equal nodal point spacing, the derivatives are replaced by the difference operators:

$$\frac{d(\ )}{dx} = \frac{(\ )_{i+1} - (\ )_{i-1}}{2\Delta}$$

and

$$\frac{d^2}{dx^2} = \frac{(\ )_{i+1} - 2(\ )_i + (\ )_{i-1}}{2\Delta^2} \tag{39}$$

where

$\Delta = L/N$

N = number of finite difference stations

The resulting system of algebraic equations is solved using the standard Potters' method [14] of forward elimination and backward substitution. Equation (34) can be solved for $f^{o''}$ after w^o has been determined.

Substituting Eq. (35) into (30) results in an equation similar to Eq. (37):

$$[D_{xx} + \frac{Q_{xx}^2}{H_{xx}}]\dot{w}_c^{o\,iv} + [\frac{2Q_{xx}}{RH_{xx}} + \frac{P_c}{2\pi R}]\dot{w}_c^{o''} + \frac{\dot{w}_c^o}{R^2 H_{xx}} = \frac{\nu A_{xy}}{2\pi E t H_{xx} R^2} - \frac{\dot{w}_c^{o''}}{2\pi R} \tag{40}$$

and

$$H_{xx}\dot{f}_c^{o''} = Q_{xx}w_c^{o''} + \frac{\dot{w}_c^o}{R} - \frac{\nu A_{xy}}{2\pi E t R} \tag{41}$$

The same imposed boundary conditions and the same solution procedure used to solve Eq. (34) is repeated.

Classical Linear Buckling Problem

Expansion of the buckling mode (Eq. 34) can now be considered. For the linear term of the expansion, the governing equations are:

$$L_D[W^{(1)}] + L_Q[F^{(1)}] + \frac{P_c}{2\pi R} W^{(1)}_{,xx} - f_c^{o''} W^{(1)}_{,yy} - w_c^{o''} F^{(1)}_{,yy} = 0$$

$$L_H[F^{(1)}] - L_Q[W^{(1)}] + w_c^{o''} W^{(1)}_{,yy} = 0 \qquad (42)$$

The solution of Eq. (42) is assumed to be of the form:

$$W^{(1)} = w^{(1)} \cos ny/R$$

$$F^{(1)} = f^{(1)} \cos ny/R \qquad (43)$$

$w^{(1)}$ and $f^{(1)}$ are determined using the previously described solution procedure. Note that $f_c^{o''}$ and $w_c^{o''}$ must also be obtained.

Initial Postbuckling

The governing equations for $W^{(2)}, F^{(2)}$ in Eq. (34) are:

$$L_D[W^{(2)}] + L_Q[F^{(2)}] = \frac{P_c}{2\pi R} W^{(2)}_{,xx} - f_c^{o''} W^{(2)}_{,yy} - w_c^{o''} F^{(2)}_{,yy}$$

$$= \frac{1}{2} \left(\frac{n}{R}\right)^2 [(f^{(1)} w^{(1)''}) + \cos (2ny/R)(f^{(1)''} w^{(1)}$$

$$+ f^{(1)} w^{(1)''} - 2f^{(1)'} w^{(1)'})] \qquad (44)$$

$$L_H[F^{(2)}] - L_Q[W^{(2)}] + w_c^{o''} W^{(2)}_{,yy} = \frac{1}{2} \left(\frac{n}{R}\right)^2 [\frac{1}{2} (w^{(1)} w^{(1)''})$$

$$+ \cos (2ny/R)(w^{(1)''} w^{(1)} - w^{(1)'} w^{(1)'})] \qquad (45)$$

The solution for $w^{(2)}$ and $F^{(2)}$ may be written as

$$W^{(2)} = w_\alpha + w_\beta \cos(2ny/R)$$
$$F^{(2)} = f_\alpha + f_\beta \cos(2ny/R) \tag{46}$$

From the boundary conditions at each end of the half length shell and Eqs. (44) and (45), the parameters $w_\alpha, w_\beta, f_\alpha$ and f_β can be determined; again using the previously described solution procedure.

Measure of Imperfection Sensitivity

In order to determine the imperfection sensitivity of the cylinder due to an imperfection having an amplitude, δ, the asymptotic formula given by Koiter [15] results in the form

$$\frac{P_s}{P_c} = 1 - 3 \sqrt[3]{\frac{-\bar{b}}{4} (\delta/t)^2} \tag{47}$$

where

$$\bar{b} = \frac{b[F_c^o*(w^{(1)},w^{(1)}) + F^{(1)}*(w_c^o,w^{(1)})]^2}{(p_c^o)^2[\dot{F}_c^o*(w^{(1)},w^{(1)}) + 2\, F^{(1)}*(\dot{w}_c^o,w^{(1)})]^2} \tag{48}$$

Thus, when the state of stress is pure membrane, $\bar{b} = b$.

2.4 Program PVRCK

This program calculates the buckling load factor and the Koiter imperfection sensitivity parameter. A circular cylindrical layered orthotropic shell is assumed. Both ring and stringer stiffeners are smeared as in the previous case. Out-of-plane bending stiffness is ignored but the torsional rigidity is included. The applied load can be any combination of axial stress, external pressure, and torsion. The boundary conditions are not specifically imposed, but simply supported conditions are approximated by the selection of the radial displacement response functions.

Khot has studied the buckling and postbuckling behavior of composite cylindrical shells [5,16]. He also has investigated the imperfection sensitivity of these shells. Several computer programs were developed by Khot. These programs have been revised to more efficiently evaluate the buckling characteristics under either separate or combined states of stress resulting from axial compression, external pressure, or torsion. Some development of the theory has already been published, but many key points remain in a preliminary and unpublished form. Some of these data have been graciously supplied by Khot.

The shallow shell strain-displacement equations and Donnell's equilibrium equations (Eqs. (2) and (3)) are again used.

The constitutive equations take the usual form:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} C & K \\ K^T & D \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \kappa \end{Bmatrix} \tag{49}$$

where

$$N = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \quad ; \quad M = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}$$

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} ; \qquad \kappa = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}$$

$$C_{\ell m} = \sum_i B^i_{\ell m}(h_{i+1} - h_i)$$

$$K_{\ell m} = \frac{1}{2} \sum_i B^i_{\ell m}(h^2_{i+1} - h^2_i)$$

$$D_{\ell m} = \frac{1}{3} \sum_i B^i_{\ell m}(h^3_{i+1} - h^3_i)$$

$B_{\ell m}$ = elastic moduli of each layer

h_i = distance to each layer from reference surface

Note: $A_{\ell m}$, $K_{\ell m}$, and $D_{\ell m}$ can be augmented by the smearing of the stiffener properties.

Solving for ε in the first equation of (49) and substituting into the second results in the semi-inverted form:

$$\begin{Bmatrix} \varepsilon \\ M \end{Bmatrix} = \begin{bmatrix} a & d^T \\ d & d^* \end{bmatrix} \begin{Bmatrix} N \\ \kappa \end{Bmatrix} \tag{50}$$

where

$$a = C^{-1}$$

$$d^T = aK$$

$$d^* = D - K^T a K$$

Using the definition of the stress function from Eq. (6), we can obtain the equilibrium equation in the form:

$$d_{12}F_{,xxxx} + (2d_{62}-d_{16})F_{,xxxy} + (d_{11}+d_{22}-2d_{66})F_{,xxyy}$$
$$+ (2d_{61}-d_{26})F_{,xyyy} + d_{21}F_{,yyyy} + F_{,xx}(w_{,yy}+\frac{1}{R}) \tag{51}$$
$$- 2F_{,xy}w_{,xy} - d^*_{11}w_{,xxxx} - 4d_{16}w_{,xxxy} - (2d^*_{12}+4d^*_{66})w_{,xxyy}$$
$$- 4d^*_{26}w_{,xyyy} - d^*_{22}w_{,yyyy} - p = 0$$

The compatibility condition is written as:

$$a_{22}F_{,xxxx} - 2a_{26}F_{,xxxy} + (2a_{12}+a_{66})F_{,xxyy} - 2a_{16}F_{,xyyy}$$
$$+ a_{11}F_{,yyyy} = - d_{12}w_{,xxxx} - (2d_{61}-d_{26})w_{,xyyy}$$
$$- d_{21}w_{,yyyy} + w^2_{,xy} - w_{,xx}w_{,yy} - \frac{w_{,xx}}{R} \tag{52}$$
$$- (2d_{62}-d_{16})w_{,xxxy} - (d_{11}+d_{22}-2d_{66})w_{,xxyy}$$

For the above two equations, the subscript notation of 1,2 and 6 correspond to the material properties in the two orthogonal directions and shear, respectively.

After a lengthy normalization procedure, Eq. (51) can be written as:

$$\kappa F_{,xxxx} + \nu F_{,xxxy} + \psi F_{,xxyy} + \xi F_{,xyyy} + \bar{\lambda} F_{,yyyy}$$
$$+ F_{,xx}w_{,yy} + \frac{1}{2}F_{,xx} - 2F_{,xy}w_{,xy} + \frac{\phi}{4}w_{,xxxx} + \nu w_{,xxxy}$$
$$+ \frac{\rho}{2}w_{,xxyy} + \zeta w_{,xyyy} + \frac{\phi}{4}w_{,yyyy} - p\phi = 0 \tag{53}$$

and Eq. (52) can be written as:

$$\frac{4}{\phi}F_{,xxxx} + \beta F_{,xxxy} + 4\alpha F_{,xxyy} - \gamma F_{,xyyy} + 4\phi F_{,yyyy}$$
$$+ \kappa w_{,xxxx} + \nu w_{,xxxy} + \psi w_{,xxyy} + \xi w_{,xyyy} + \bar{\lambda} w_{,yyyy} \tag{54}$$
$$+ 2w_{,xx} = w^2_{,xy} - w_{,xx}w_{,yy}$$

where

$$\kappa = \frac{d_{12}}{a_{22}} \left(\frac{a_{11}}{d^*_{22}}\right)^{1/2} \frac{1}{\phi}$$

$$\xi = \frac{2d_{61}-d_{26}}{(a_{22}d^*_{22})^{1/2}} \left(\frac{a_{22}}{a_{11}}\right)^{1/4} (\phi)^{1/2}$$

$$\gamma = 8 \frac{a_{12}}{(a^3_{11}a_{22})^{1/4}} (\phi)^{1/2}$$

$$\beta = 8 \frac{a_{26}}{(a_{11}a^3_{22})^{1/4}} (\phi)^{-1/2}$$

$$\alpha = \frac{2a_{12}+a_{66}}{(a_{11}a_{22})^{1/2}}$$

$$\eta = \frac{d^*_{16}}{(d^{*3}_{11}d^*_{22})^{1/4}} \phi$$

$$\zeta = \frac{d^*_{26}}{(d^*_{11}d^{*3}_{22})^{1/4}} \phi$$

$$\nu = \frac{(2d_{62}-d_{16})}{(a_{22}d^*_{22})^{1/2}} \left(\frac{a_{11}}{a_{22}}\right)^{1/4} (\phi)^{-1/2}$$

$$\psi = \frac{d_{11}+d_{22}-2d_{66}}{(a_{22}d^*_{22})^{1/2}}$$

$$\phi = \left(\frac{a_{11}d^*_{11}}{a_{22}d^*_{22}}\right)^{1/2}$$

$$\overline{\lambda} = \frac{d_{21}}{(a_{11}d^*_{22})^{1/2}} \phi$$

Eqs. (53) and (54) are now of the form that the critical load parameter, λ_c, and the imperfection sensitivity parameter of the shell can be derived.

For the present case a solution can be obtained with Eq. (34) rewritten as:

$$F = \lambda_c F^o + \delta F^{(1)} + \delta^2 F^{(2)} + \ldots$$

$$w = \qquad \delta w^{(1)} + \delta^2 w^{(2)} + \ldots$$

Substitution of these equations into (53) and (54) and collection of like powers of δ results in:

$$L_1[w^{(1)}] - L_2[F^{(1)}] = \lambda_c[F^o_{,xx}w^{(1)}_{,yy} + F^o_{,yy}w^{(1)}_{,xx} - 2F^o_{,xy}w^{(1)}_{,xy}] \quad (55a)$$

$$L_2[w^{(1)}] + L_3[F^{(2)}] = 0.0 \quad (55b)$$

and

$$L_1[w^{(2)}] - L_2[F^2] - \lambda_c[F^o_{,xx}w^{(2)}_{,yy} + F^o_{,yy}w^{(2)}_{,xx} - 2F^o_{,xy}w^{(2)}_{xy}]$$
$$= F^{(1)}_{,xx}w^{(1)}_{,yy} + F^{(1)}_{,yy}w^{(1)}_{,xx} - 2F^{(1)}_{,xy}w^{(1)}_{,xy} \quad (55c)$$

$$L_2[w^{(2)}] + L_3\ [F^{(2)}] = (w^{(1)}_{,xy})^2 - w^{(1)}_{,xx}w^{(1)}_{,yy} \quad (55d)$$

where

$$L_1[\ ] = \frac{\phi}{4}[\ ]_{,xxxx} + \eta[\ ]_{,xxxy} + \frac{\rho}{2}[\ ]_{,xxyy} + \zeta[\ ]_{,xyyy} + \frac{\phi}{4}[\ ]_{,yyyy}$$

$$L_2[\ ] = \kappa[\ ]_{,xxxx} + \nu[\ ]_{,xxxy} + \psi[\ ]_{,xxyy} + \xi[\ ]_{,xyyy}$$
$$+ \bar{\lambda}[\ ]_{,yyyy} + 2[\ ]_{,xx}$$

$$L_3[\ ] = \frac{4}{\phi}[\ ]_{,xxxx} - \beta[\ ]_{,xxxy} + 4\alpha[\ ]_{,xxyy}$$
$$- \gamma[\ ]_{,xyyy} + 4\phi[\ ]_{,yyyy}$$

Neglecting the nonlinear coupling, λ_c can be determined for the case of combined axial compression, external pressure and

torsion. It is assumed that the classical radial displacement under torsion can be of the form:

$$w = h \sin \frac{m\pi x}{\ell} \cos \frac{n}{r} (y-\tau x) \qquad (56)$$

where

$$h = \frac{t}{(a_{22}d^*_{22})^{1/2}}$$

$$r = \frac{R}{(4R^2 d^*_{22} a_{22})^{1/4}}$$

$$\ell = \frac{L}{(4R^2 d^*_{11} a_{22})^{1/4}}$$

τ is an integer

If the notation

$$M = \frac{m\pi}{\ell} + \frac{n}{r}\tau$$

$$N = \frac{n}{r}$$

$$P = \frac{m\pi}{\ell} - \frac{n}{r}\tau$$

is used, then the classical radial deflection for the initial buckling mode can be written:

$$w^{(1)} = \frac{h}{2} \sin (Mx-Ny) + \sin (Px + Ny) \qquad (57)$$

or $$w^{(1)} = \frac{h}{2} (A_1+A_2)$$

It is assumed that the Airy's stress function can be expressed in a similar form:

$$F^{(1)} = F_1A_1 + F_2A_2 \qquad (58)$$

From the compatibility equation (54) and the definition of $w^{(1)}$ and $F^{(1)}$, we obtain:

$$F_1 = -\frac{h}{2}\frac{\kappa M^4 - \nu M^3 N + \psi M^2 N^2 - \xi M N^3 + \bar{\lambda} N^4 - 2M^2}{\frac{4}{\phi} M^4 + \beta M^3 N + 4\alpha M^2 N^2 + \gamma M N^3 + 4\phi N^4}$$

$$= -\frac{h}{2}\frac{T_3}{T_5} \tag{59a}$$

$$F_2 = -\frac{h}{2}\frac{\kappa P^4 + \nu P^3 N + \psi P^2 N^2 + \xi P N^3 + \bar{\lambda} N^4 - 2P^2}{\frac{4}{\phi} P^4 - \beta P^3 N + 4\alpha P^2 N^2 - \gamma P N^3 + 4\phi N^4}$$

$$= -\frac{h}{2}\frac{T_4}{T_6} \tag{59b}$$

From Eq. (53) we obtain:

$$T_1A_1 + T_2A_2 - T_3F_1A_1 - T_4F_2A_2 = -\lambda_c \frac{h}{2}[N_x^o(M^2A_1 + P^2A_2)$$

$$+ N_y^o N^2(A_1+A_2) + 2N_{xy}^o(-NMA_1 + NPA_2)] \tag{60}$$

where

$$T_1 = \frac{h}{2}\left(\frac{\phi}{4} M^4 - \eta P^3 N + \frac{\rho}{2} M^2 N^2 - \zeta M N^3 + \frac{\phi}{4} N^4\right)$$

$$T_2 = \frac{h}{2}\left(\frac{\phi}{4} P^4 + \eta P^3 N + \frac{\rho}{2} P^2 N^2 + \zeta P N^3 + \frac{\phi}{4} P^4\right)$$

Collecting like terms, we can now solve the critical load factor:

$$\lambda_c = \frac{-[T_1 + T_2 + \frac{T_3^2}{T_5} + \frac{T_4^2}{T_6}]}{N_x^o(M^2+P^2) + 2N_y^o N^2 + 2N_{xy}^o N(P-M)} \tag{61}$$

where:

$$N_x^o = \frac{\bar{N}_x R}{2} \left(\frac{a_{11}}{d_{22}^*}\right)^{1/2} \quad ; \quad N_y^o = \frac{\bar{N}_y R}{2} \left(\frac{a_{22}}{d_{22}^*}\right)^{1/2} \quad ; \quad N_{xy}^o = \left(\frac{(a_{11}a_{22})^{1/4}}{4(d_{22}^*)^{1/2}}\right) \bar{N}_{xy} R$$

are the stress resultants calculated from the applied loads $\bar{N}_x, \bar{N}_y$ and $\bar{N}_{xy}$ and are assumed to be a pure membrane state.

In order to evaluate the coefficients a and b of the expansion of Eq. (1), Eq. 32 must be used with Eqs. (57), (59a), and (59b). The coefficient, a, has been determined to be zero from consideration of the periodicity of the circumferential displacement.

With $F^{(1)}$ and $w^{(1)}$ determined, the postbuckling $w^{(2)}$ and $F^{(2)}$ are assumed of the form:

$$w^{(2)} = \sum_{i=odd} \alpha_i \sin \frac{i\pi x}{\ell} + \frac{1}{2} \cos \frac{2ny}{r} \sum_{i=odd} \gamma_i (\sin M_i x + \sin P_i x) + \frac{1}{2} \sin \frac{2ny}{r} \sum_{i=odd} \gamma_i (\cos P_i x - \cos M_i x) \tag{62}$$

$$F^{(2)} = \sum_{i=odd} \beta_i \sin \frac{i\pi x}{\ell} + \frac{1}{2} \cos \frac{2ny}{r} \sum_{i=odd} \delta_i (\sin M_i x + \sin P_i x) + \frac{1}{2} \sin \frac{2ny}{r} \sum_{i=odd} \delta_i (\cos P_i x - \cos M_i x)$$

where

$$M_i = \frac{i\pi}{\ell} + \frac{2ny}{r}$$

$$P_i = \frac{i\pi}{\ell} - \frac{2ny}{r}$$

It remains to determine the coefficients α_i, β_i, γ_i and δ_i. To obtain relationships for these coefficients, Eqs. (59a), (59b) and (62) are substituted into Eqs. (58) and (55c) to obtain:

$$
\begin{aligned}
&\frac{\phi}{4}[\sum \alpha_i \sin(\frac{i\pi x}{\ell}) + \frac{1}{2} \mathrm{CSR} \sum \gamma_i \Lambda_4 + \frac{1}{2} \mathrm{SNR} \sum \gamma_i \Omega_4] \\
&\quad + \eta[\frac{1}{2}(\frac{2n}{R}) \mathrm{SNR} \sum \gamma_i \Omega_3 + \frac{1}{2} \mathrm{CSR} \sum \gamma_i \Lambda_3] \\
&\quad + \frac{\rho}{2}[\frac{1}{2}(\frac{2n}{R}(^2 \mathrm{CSR} \sum \gamma_i \Lambda_2 - \frac{1}{2}(\frac{2n}{R})^2 \mathrm{SNR} \sum \gamma_i \Omega_2] \\
&\quad + \zeta[\frac{2}{2}(\frac{2n}{R})^3 \mathrm{SNR} \sum \gamma_i \Omega_1 + \frac{1}{2}(\frac{2n}{R})^3 \mathrm{CSR} \sum \gamma_i \Lambda_1] \\
&\quad + \frac{\phi}{4}[\frac{1}{2}(\frac{2n}{R})^4 \mathrm{CSR} \sum \gamma_i \Lambda_o + \frac{1}{2}(\frac{2n}{R})^4 \mathrm{SNR} \sum \gamma_i \Omega_o] \qquad (63) \\
&\quad - \kappa[\sum \beta_i (\frac{i\pi}{\ell})^4 \sin(\frac{i\pi x}{\ell}) + \frac{1}{2} \mathrm{CSR}) \sum \delta_i \Lambda_4 + \frac{1}{2} \mathrm{SNR} \sum \delta_i \Omega_4] \\
&\quad - \nu[\frac{1}{2}(\frac{2n}{R}) \mathrm{SNR} \sum \delta_i \Omega_3 + \frac{1}{2}(\frac{2n}{R}) \mathrm{CSR} \sum \delta_i \Lambda_3] \\
&\quad + \Psi[\frac{1}{2}(\frac{2n}{R})^2 \mathrm{CSR} \sum \delta_i \Lambda_2 - \frac{1}{2}(\frac{2n}{R})^2 \mathrm{SNR} \sum \delta_i \Omega_2] \\
&\quad - \xi[\frac{1}{2}(\frac{2n}{R})^3 \mathrm{SNR} \sum \delta_i \Omega_1 - \frac{1}{2}(\frac{2n}{R})^3 \mathrm{CSR} \sum \delta_i \Lambda_1] \\
&\quad - \overline{\lambda}[\frac{1}{2}(\frac{2n}{R})^4 \mathrm{CSR} \sum \delta_i \Lambda_o + \frac{1}{2}(\frac{2n}{R})^4 \mathrm{SNR} \sum \delta_i \Omega_o] \\
&\quad - 2[\sum \alpha_i (\frac{i\pi}{\ell})^2 + \frac{1}{2} \mathrm{CSR} \sum \delta_i \Lambda_2 + \frac{1}{2} \mathrm{SNR} \sum \delta_i \Omega_2] \\
&\quad - \lambda_c[N_y^o\{-\frac{1}{2}(\frac{2n}{R})^2 \mathrm{CSR} \sum \gamma_i \Lambda_p - \frac{1}{2}(\frac{2n}{R})^2 \mathrm{SNR} \sum \gamma_i \Omega_o\} \\
&\qquad + N_x^o\{- \sum \alpha_i (\frac{i\pi}{\ell})^2 \sin \frac{i\pi x}{\ell} \\
&\qquad\qquad - \frac{1}{2} \mathrm{CSR} \sum \gamma_i \Lambda_2 + \frac{1}{2} \mathrm{SNR} \sum \gamma_i \Omega_2\} \\
&\qquad + 2N_{xy}^o\{- \frac{1}{2}(\frac{2n}{R}) \mathrm{SNR} \sum \gamma_i \Omega_1 \\
&\qquad\qquad + \frac{1}{2} (\frac{2n}{R}) \mathrm{CSR} \sum \gamma_i \Lambda_1 \}] \\
&\quad = - \frac{h^2}{4}[\frac{T_3}{T_5} + \frac{T_4}{T_6}] N^2[M+P]^2 A_1 A_2
\end{aligned}
$$

where

$$\mathrm{SNR} = \sin(\tfrac{2ny}{R})$$

$$\mathrm{CSR} = \cos(\tfrac{2ny}{R})$$

$$\Lambda_o = \sin(M_i x) + \sin(P_i x)$$

$$\Lambda_1 = M_i \sin(M_i x) + \sin(P_i x)$$

$$\Lambda_2 = M_i^2 \sin(M_i x) + P_i^2 \sin(P_i x)$$

$$\Lambda_3 = -M_i^3 \sin(M_i x) + P_i^3 \sin(P_i x)$$

$$\Lambda_4 = M_i^4 \sin(M_i x) + P_i^4 \sin(P_i x)$$

$$\Omega_o = \cos(M_i x) + \cos(M_i x)$$

$$\Omega_1 = -M_i \cos(M_i x) + P_i \cos(P_i x)$$

$$\Omega_2 = M_i^2 \cos(M_i x) - P_i^2 \cos(P_i x)$$

$$\Omega_3 = M_i^3 \cos(M_i x) + P_i^3 \cos(P_i x)$$

$$\Omega_4 = -M_i^4 \cos(M_i x) + P_i^4 \cos(P_i x)$$

Also, Eqs. (59a), (59b) and (62) are substituted into Eq. (55d) to obtain:

$$\begin{aligned}
&\kappa[\sum \alpha_i(\tfrac{i\pi}{\ell})^4 \sin(\tfrac{i\pi x}{\ell}) + \tfrac{1}{2}\,\mathrm{CSR} \sum \gamma_i \Lambda_4 + \tfrac{1}{2} \sum \gamma_i \Omega_4] \\
&\quad + \nu[\tfrac{1}{2}(\tfrac{2n}{R})\,\mathrm{SNR} \sum \gamma_i \Omega_3 + \tfrac{1}{2}(\tfrac{2n}{R})\,\mathrm{CSR} \sum \gamma_i \Lambda_3] \\
&\quad + \Psi[\tfrac{1}{2}(\tfrac{2n}{R})^2\,\mathrm{CSR} \sum \gamma_i \Lambda_2 - \tfrac{1}{2}(\tfrac{2n}{R})^2\,\mathrm{SNR} \sum \gamma_i \Omega_2] \\
&\quad + \xi[\tfrac{1}{2}(\tfrac{2n}{R})^3\,\mathrm{SNR} \sum \gamma_i \Omega_1 - \tfrac{1}{2}(\tfrac{2n}{R})^3\,\mathrm{CSR} \sum \gamma_i \Lambda_1] \\
&\quad + \bar{\lambda}[\tfrac{1}{2}(\tfrac{2n}{R})^4\,\mathrm{CSR} \sum \gamma_i \Lambda_o + \tfrac{1}{2}(\tfrac{2n}{R})^4 \sum \gamma_i \Omega_o] \\
&\quad + 2[-\sum \alpha_i(\tfrac{i\pi}{\ell})^2 \sin(\tfrac{i\pi x}{\ell}) - \tfrac{1}{2}\,\mathrm{CSR} \sum \gamma_i \Lambda_2 + \tfrac{1}{2}\,\mathrm{SNR} \sum \gamma_i \Omega_2]
\end{aligned} \tag{64}$$

$$+ \frac{4}{\phi}[\sum \beta_i \ (\frac{i\pi}{\ell})^4 \sin (\frac{i\pi x}{\ell}) + \frac{1}{2} \text{CSR} \sum \delta_i \Lambda_4 + \frac{1}{2} \text{SNR} \sum \delta_i \Omega_4]$$

$$- \beta[\frac{1}{2}(\frac{2n}{R}) \text{SNR} \sum \delta_i \Omega_3 + \frac{1}{2}(\frac{2n}{R}) \text{CSR} \sum \delta_i \Lambda_3]$$

$$+4\alpha[\frac{1}{2}(\frac{2n}{R})^2 \text{CSR} \sum \delta_i \Lambda_2 - \frac{1}{2}(\frac{2n}{R})^2 \text{SNR} \sum \delta_i \Omega_2]$$

$$- \gamma[\frac{1}{2} (\frac{2n}{R})^3 \text{SNR} \sum \delta_i \Omega_1 - \frac{1}{2} (\frac{2n}{R})^3 \text{CSR} \sum \delta_i \Lambda_1]$$

$$+4\phi[\frac{1}{2} (\frac{2n}{R}) \text{CSR} \sum \delta_i \Lambda_o + \frac{1}{2} (\frac{2n}{R})^4 \text{SNR} \sum \delta_i \Omega_o]$$

$$= - \frac{h^2}{4} N^2[M+P]^2 A_1 A_2$$

Multipling Eq. (63) by $\sin \frac{i\pi x}{\ell}$ and integrating between $\int_0^{\ell} \int_0^{2\pi R}$ we obtain:

$$\frac{\phi}{4}[\alpha_i(\frac{i\pi}{\ell})^4] - \kappa[\beta_i(\frac{i\pi}{\ell})^4 - 2\beta_i(\frac{i\pi}{\ell})^2] + \lambda_c N_x^o \alpha_i(\frac{i\pi}{\ell})^2$$

$$= \frac{h^2}{4}(\frac{T_3}{T_5} + \frac{T_4}{T_6})N^2 \frac{4\pi m^2}{\ell^2} \frac{2i}{(i^2-4m^2)} \qquad (65)$$

Multipling Eq. (64) by $\sin \frac{i\pi x}{\ell}$ and integrating, we obtain:

$$\kappa\alpha_i(\frac{i\pi}{\ell})^4 - 2\alpha_i(\frac{i\pi}{\ell})^2 + \frac{4}{\phi} \beta_i(\frac{i\pi}{\ell})^4$$

$$= h^2N^2 \frac{m^2\pi}{\ell^2} \frac{2i}{(i^2-4m^2)} \qquad (66)$$

Eliminating β_i we obtain:

$$\alpha_i = \frac{\frac{2h^2N^2m^2}{(i^2-4m^2)} \frac{\pi i}{\ell^2} \{(\frac{T_3}{T_5} + \frac{T_4}{T_6}) \frac{4}{\phi} + \kappa - 2(\frac{\ell}{i\pi})^2\}}{(\frac{i\pi}{\ell})^4 + \lambda_c N_x^o \frac{4}{\phi} (\frac{i\pi}{\ell})^2 + \{\kappa(\frac{i\pi}{\ell})^2 - 2\}^2} \qquad (67)$$

and from Eq. (66)

$$\beta_i = \phi[\frac{h^2N^2m^2}{(i^2-4m^2)} \frac{\pi i}{2\ell^2} (\frac{\ell}{i\pi})^4 - \alpha_i\{\frac{\kappa}{4} - \frac{1}{2} (\frac{\ell}{i\pi})^2\}] \qquad (68)$$

Multiplying Eq. (63) by

$$\frac{1}{2} \cos \frac{2ny}{R} [\sin M_i x + \sin P_i x]$$

$$+ \frac{1}{2} \sin \frac{2ny}{R} [\cos P_i x - \cos M_i x]$$

and integrating, results in:

$$\begin{aligned}
\gamma_i[&\tfrac{\phi}{16}(M_i^4+P_i^4) - \tfrac{\eta}{4}(\tfrac{2n}{R})(M_i^3-P_i^3) + \tfrac{\rho}{8}(\tfrac{2h}{R})^2(M_i^2+P_i^2) - \tfrac{\zeta}{4}(\tfrac{2n}{R})^3(M_i-P_i) \\
&+ \tfrac{\phi}{8}(\tfrac{2n}{R})^4] - \delta_i[\tfrac{\kappa}{4}(M_i^4+P_i^4) - \tfrac{\nu}{4}(\tfrac{2n}{R})(M_i^3-P_i^3) + \tfrac{\psi}{4}(\tfrac{2n}{R})^2(M_i^2-P_i^2) \\
&- \tfrac{\xi}{4}(\tfrac{2n}{R})^3(M_i-P_i) + \tfrac{\overline{\lambda}}{2}(\tfrac{2n}{R})^4 - \tfrac{1}{2}(M_i^2+P_i^2)] + \lambda_c \tfrac{\gamma_i}{2}[N_y^o(\tfrac{2n}{R})^2 \\
&+ \tfrac{N_x^o}{2}(M_i^2+P_i^2) - N_{xy}^o(\tfrac{2n}{R})(M_i-P_i)] = \tfrac{h^2}{4}(\tfrac{T_3}{T_5} + \tfrac{T_4}{T_6}) \frac{N^2(M+P)^2}{i\pi}
\end{aligned} \tag{69}$$

Multiplying Eq. (64) by the same factor and integrating, results in:

$$\begin{aligned}
\gamma_i[&\tfrac{\kappa}{4}(M_i^4+P_i^4) - \tfrac{\nu}{4}(\tfrac{2n}{R})(M^3-P_i{}^3) + \tfrac{\psi}{4}(\tfrac{2n}{R})^2(M_i^2+P_i^2) - \tfrac{\xi}{4}(\tfrac{2n}{R})^3(M_i-P_i) \\
&+ \tfrac{\overline{\lambda}}{2}(\tfrac{2n}{R})^4 - \tfrac{1}{2}(M_i^2+P_i^2)] + \delta_i[\tfrac{1}{\phi}(M_i^4+P_i^4) + \tfrac{\beta}{4}(M_i^3-P_i^3) \\
&+ \alpha(\tfrac{2n}{R})^2(M_i^2+P_i^2) + \tfrac{\beta}{4}(\tfrac{2n}{R})(M_i^3-P_i^3) + \tfrac{\gamma}{4}(\tfrac{2n}{R})^3(M_i-P_i) \\
&+ 2\phi(\tfrac{2n}{R})^4] = -\tfrac{h^2}{4} N^2 \frac{(M+P)^2}{i\pi}
\end{aligned} \tag{70}$$

Solving for δ_i in Eq. (70) and substituting into Eq. (69), we obtain:

$$\gamma_i = \frac{\frac{-4h^2N^2m^2\pi^2}{\ell^2} (\frac{1}{i\pi})\{(\frac{T_3}{T_5} + \frac{T_4}{T_6}) T_9 + T_8\}}{T_7T_9 + T_8^2} \tag{71}$$

$$\delta_i = -\left[\frac{4h^2N^2m^2\pi^2}{\ell^2}\left(\frac{1}{i\pi}\right) + \gamma_i T_8\right]\frac{1}{T_9} \qquad (72)$$

where

$$T_7 = \frac{\phi}{4}(M_i^4+P_i^4) - \eta\left(\frac{2n}{R}\right)(M_i^3-P_i^3) + \frac{\rho}{2}\left(\frac{2n}{R}\right)^2(M_i^2+P_i^2)$$
$$- \zeta\left(\frac{2n}{R}\right)^3(M_i-P_i) + \frac{\phi}{4}\left(\frac{2n}{R}\right)^4 + \lambda_c[N_x^o\{M_i^2+P_i^2\}$$
$$+ 2N_y^o\left(\frac{2n}{R}\right)^2 - 2N_{xy}^o\{M_i-P_i\}\frac{2n}{R}]$$
$$T_8 = \kappa(M_i^4+P_i^4) - \nu\left(\frac{2n}{R}\right)(M_i^3-P_i^3) + \psi\left(\frac{2n}{R}\right)^2(M_i^2+P_i^2)$$
$$- \xi\left(\frac{2n}{R}\right)^3(M_i-P_i) + 2\bar{\lambda}\left(\frac{2n}{R}\right)^4 - 2(M_i^2-P_i^2)$$

$$T_9 = \frac{4}{\phi}(M_i^4+P_i^4) + \beta\left(\frac{2n}{R}\right)(M_i^3-P_i^3) + 4\alpha\left(\frac{2n}{R}\right)^2(M_i^2+P_i^2)$$
$$+ \gamma\left(\frac{2n}{R}\right)^3(M_i-P_i) + 8\phi\left(\frac{2n}{R}\right)^4$$

Equation (32) can be rewritten as:

$$b = \{2\iint[F^{(1)}_{,xx}W^{(1)}_{,y}W^{(2)}_{,y} + F^{(1)}_{,yy}W^{(1)}_{,x}W^{(2)}_{,x}$$
$$- F^{(1)}_{,xy}\{W^{(1)}_{,x}W^{(2)}_{,y} + W^{(1)}_{,y}W^{(2)}_{,x}\}]dxdy + \iint[F^{(2)}_{,xx}(W^{(1)}_{,y})^2$$
$$+ F^{(2)}_{,yy}(W^{(1)}_{,x})^2 - 2F^{(2)}_{,xy}W^{(1)}_{,x}W^{(1)}_{,y}]dxdy\} \quad /$$
$$- \lambda_c\iint[F^o_{,xx}(W^{(2)}_{,y})^2 + F^o_{,yy}(W^{(1)}_{,x})^2$$
$$- 2F^o_{,xy}W^{(1)}_{,x}W^{(1)}_{,y}]dxdy \qquad (73)$$

Substituting $W^{(1)}$, $F^{(1)}$, $W^{(2)}$, and $F^{(2)}$ into Eq. (73) results in the following expressions:

$$b = \frac{4N^2 m^2 \pi}{\lambda_c \ell^2} \frac{T_{10}}{T_{11}} \tag{74}$$

where

$$T_{10} = \left\{\frac{T_3}{T_5} + \frac{T_4}{T_6}\right\} \left\{ \sum_{i=odd} \alpha_i \frac{2i}{i^2-4m^2} - \sum_{i=odd} \frac{\gamma_i}{i} \right\}$$

$$- \sum \beta_i \frac{2i}{i^2-4m^2} + \sum \frac{\delta_i}{i}$$

$$T_{11} = N_x^o \{(\frac{m\pi}{\ell})^2 + N^2 \tau^2\} + N_y^o N^2 - 2N_{xy}^o N^2 \tau$$

With an imperfection amplitude in a given harmonic δ_m specified, the knockdown factor (λ_s/λ_c) can then be determined by

$$(1 - \lambda_s/\lambda_c)^{3/2} = \frac{3}{2} (\lambda_s/\lambda_c) \sqrt{-3b} \left|\frac{\delta_m}{t}\right| \tag{75}$$

2.5 Program PVRCA

In the previous sections, the computer programs are rather limited in scope with respect to the assumed imperfection modes. At most, just two imperfection modes could be considered simultaneously. In order to develop a correlation to experimental data, general imperfection data must be derived from specific measured geometric data. Arbocz has written a procedure that accounts for the effects of a general imperfection when a cylinder stiffened by rings or by stringers is subjected to an axial load. Further, the effects of nonlinear prebuckling and boundary conditions are incorporated into the procedure.

The in-plane and out-of-plane bending of the stiffeners are considered in the analysis. A general imperfection geometry can be specified. Either load or end-shortening increments can be considered.

A detailed description of the development of governing equations for the multimode analysis is given in Ref. [17]. Arbocz and Babcock [6] give a synopsis of the procedure and show how the quasilinearization method of Newton is used to solve the Donnell equations (Eq. 7 and 8). The solution to these equations are based upon the approximation:

$$
\begin{aligned}
W_{m+1} &= W_m + \delta W_m \\
F_{m+1} &= F_m + \delta F_m
\end{aligned}
\tag{76}
$$

where

W_m, F_m = mth approximation to the solution

$\delta W_m, \delta F_m$ = correction to the mth approximation

Substituting into Eqs. (7) and (8) and neglecting squared terms yields:

$$L_H(\delta F_m) - L_Q(\delta W_m) + \delta W_{m,xx}/R + L_{NL}(W_m+\hat{W}, \delta W_m) = - E_m^{(1)}$$

$$L_Q(\delta F_m) + L_D(\delta W_m) - \delta F_{m,xx}/R - L_{NL}(F_m, \delta W_m) \tag{77}$$

$$- L_{NL}(W_m+\hat{W}, \delta F_m) = - E_m^{(2)}$$

where

$$L_{NL}(S,T) = S_{,xx}T_{,yy} - 2S_{,xy}T_{,xy} + S_{,yy}T_{,xy}$$

$$E_m^{(1)} = L_H(F_m) - L_Q(W_m) + W_m{,xx}/R + \frac{L_{NL}}{2}(W_m,W_m+2\hat{W})$$

$$E_m^{(2)} = L_Q(F_m) + L_D(W_m) - F_m{,xx}/R - L_{NL}(F_m,W_m+\hat{W})$$

The radial imperfection is assumed to be represented by:

$$\hat{w} = t \sum_{i=1}^{N_1} \hat{w}_{io} \cos i\bar{x} + t \sum_{k,\ell=1}^{N_2} \sum \hat{w}_{k\ell} \sin k\bar{x} \cos \ell\bar{y}$$

$$+ t \sum_{k,\ell=1}^{N_3} \sum \hat{w}_{k\ell} \sin k\bar{x} \sin \ell\bar{y} = t\, L_w[\hat{w}] \tag{78}$$

where

$$\bar{x} =- \pi x/L$$

$$\bar{y} = y/R = \theta$$

Similar solution forms can be made for the four unknowns of Eqs. (76) and (77).

$$w_m = - \frac{t\nu}{c} \frac{\bar{H}_{xx}}{1+\alpha_r} + t\, L_w[w]$$

$$\delta w_m = t\, L_w[\delta w]$$

TABLE 3. RING-STIFFENED SHELLS

Parameter	Case Number 1A	12	21	24
R	3.929	3.929	3.891	3.891
H	0.0165	0.0173	0.0185	0.0223
ℓ	5.00	11.00	11.125	8.90
No Rings	9	22	22	18
d_r	0.50	0.50	0.50	0.50
e_r	-0.0193	0.0221	0.0226	0.0241
$A_r \times 10^2$	0.175	0.391	0.3072	0.295
$I_r \times 10^6$	0.0877	0.271	0.2156	0.1961
$J_r \times 10^6$	0.807	0.90	0.69	0.62
δ	0.0	0.0	0.00187	0.00195
P_{cr} Test	486	538	480	771
n_{cr}	8	8	8	8
PVRCB	470.3	562.6	531.2	705.7
n_{cr}	7	10	11	11
PVRCH	400.5	505	544.2	652.4
n_{cr}	7	9	12	11
PVRCK	469	508	531.7	776.5
n_{cr}	12	9	10	10

TABLE 4. STRINGER-STIFFENED SHELLS

Parameter	Case Number 101	105	121B	124A
R	3.694	3.703	3.759	3.762
H	0.0349	0.0194	0.0227	0.0180
ℓ	10.89	10.89	5.07	2.12
No. Stringers	36	36	36	36
d_s	0.649	0.649	0.666	0.666
e_s	0.0787	0.071	0.0521	0.0498
$A_s \times 10^2$	2.60	2.60	12.13	12.13
$I_s \times 10^6$	46.33	46.33	6.73	6.73
$J_s \times 10^6$	78.95	78.95	17.67	17.67
δ	0.0	0.0	0.0029	0.00182
P_{cr} Test	7026	3980	2640	3360
n_{cr}	-	-	6	7
PVRCH	5370	2664	3139	7371
n_{cr}	7	7	6	6
PVRCK	4700	2862	2120	2867
n_{cr}	6	4	6	6

the critical buckling load. If too small a step is taken, no solution may result. If too large a load step is taken, a root may be missed.

Programs PVRCB, PVRCH and PVRCK all give reasonably good results for a ring-stiffened cylinder. The maximum spread between the experimental results and the three programs' predictions is 18 percent. For the stringer stiffened shells, a much greater deviation is observed between the reported experimental results and the two programs PVRCH and PVRCK. It appears that the effect of boundary conditions is more critical for this class of shell than the ring-stiffened shell, as would be anticipated. PVRCK appears to be the more conservative and consistent program.

As a demonstration of the output from each program, Case 24, a ring-stiffened cylinder, is used. In Vol. II (User's Manual), the CRT displays are given for Programs PVRCB, PVRCH, and PVRCK, respectively. The results from PVRCB require no further explanation. From PVRCH, a minimum load of 862.2 lbs at a response mode of n_{cr} = 11 is obtained. In order to obtain the minimum load we must use the relation:

$$P_{cr} = P_c(1-3(2)^{-1/3}\sqrt[3]{-b}\;(\delta/t)^{2/3})$$

$$= 862.2(1-1.89(.2791)^{1/3}(\frac{.00195}{.0223})^{2/3})$$

$$= 652.4 \text{ lbs.}$$

From PVRCK a minimum stress resultant of 31.76 lb/in was predicted. The minimum axial load is

$$P_{cr} = N_x 2\pi R$$

$$= 31.76(2)\pi(3.891)$$

$$= 776.5 \text{ lbs.}$$

The experimental critical buckling load was 771 lbs.

The program PVRCA cannot be run, at present, on an interactive basis. Therefore, no CRT output is possible with the present machine configuration. If larger core limits were available, interactive sessions could easily be obtained. This program uses imperfection data from existing shells and then determines the critical buckling load. Without prior knowledge of what response modes are important, the selection of the major responding modes is by pure chance. For example, the data for the first ring-stiffened cylinder (1A) was used with imperfections of the first two terms in Eq. (78) being 0.00001 for modes (9,0),(18,0), (27,0),(9,7),(9,8),(9,10),(9,11) and (9,12). The estimated critical buckling load is λ = 1.234 or 523 lbs. For Case 24, where the imperfection is known to exist (ξ = 0.00195), the predicted critical buckling load is 774 lbs with the same responding modes.

A further demonstration of the adequacy of the program PVRCA is represented by an example of the third category. From Arbocz's thesis [18], the description of an unstiffened imperfect cylindrical shell is given (see Tables 5 and 6). The form of the imperfection is:

$$
\begin{aligned}
\bar{w}(x,y) = & \sum_{m=0}^{N} \sum_{n=0}^{N} A_{mn} \cos m \frac{y}{R} \cos \frac{2n\pi x}{L} \\
& + \sum_{m=1}^{N} \sum_{n=0}^{N} B_{mn} \sin m \frac{y}{R} \cos \frac{2n\pi x}{L} \\
& + \sum_{m=0}^{N} \sum_{n=1}^{N} C_{mn} \cos m \frac{y}{R} \sin \frac{2n\pi x}{L} \\
& + \sum_{m=1}^{N} \sum_{n=1}^{N} D_{mn} \sin m \frac{y}{R} \sin \frac{2n\pi x}{L}
\end{aligned}
$$

where

TABLE 5. A7 Fourier Cosine Coefficients $\begin{bmatrix} A_{mn} \\ C_{mn} \end{bmatrix}$

N \ M	0	1/2	1	3/2	2	5/2	3	7/2
0		0.068	0.068					
1								
2	-0.402	-0.457 -0.518		-0.049 -0.155		 -0.085	 -0.103	 -0.054
3	-0.456	-0.601 -0.586	 -0.045			 -0.099		 -0.062
4	0.103	0.048 0.136						
5	0.193	0.073 0.242	 0.056					
6	0.114	 0.145						
7		-0.047						
8		-0.050						
9		 0.051						
10	-0.048	 -0.073						

TABLE 6. A7 Fourier Sine Coefficients $\begin{bmatrix} B_{mn} \\ D_{mn} \end{bmatrix}$

N \ M	0	1/2	1	3/2	2	5/2	3	7/2
0								
1								
2	-0.241	-0.348 -0.308	-0.271	-0.100	-0.126	-0.054	-0.079	
3	0.150	0.189		0.066				
4	0.504	0.129 0.644	0.100	0.200	0.048	0.112		
5	0.128	0.167		0.047				
6	-0.165	-0.149 -0.200	-0.108		-0.063	-0.047		
7	-0.074	-0.075		-0.056				
8		0.069	0.049					
9								
13		-0.048						

$$A_{mn} = \frac{2}{\pi L}\int_0^L \int_0^{2\pi R} \overline{w}(x,y) \cos m \frac{y}{R} \cos \frac{2n\pi x}{L} \, dy \, dx$$

$$B_{mn} = \frac{2}{\pi L}\int_0^L \int_0^{2\pi R} \overline{w}(x,y) \sin m \frac{y}{R} \cos \frac{2n\pi x}{L} \, dy \, dx$$

$$C_{mn} = \frac{2}{\pi L}\int_0^L \int_0^{2\pi R} \overline{w}(x,y) \cos m \frac{y}{R} \sin \frac{2n\pi x}{L} \, dy \, dx$$

$$D_{mn} = \frac{2}{\pi L}\int_0^L \int_0^{2\pi R} \overline{w}(x,y) \sin m \frac{y}{R} \sin \frac{2n\pi x}{L} \, dy \, dx$$

A stiffened cylindrical shell (A7) was used with the following properties:

Radius	=	4.003 in.
Length	=	8.00 in.
Wall Thickness	=	0.004494 in.
E	=	15.1×10^6 psi.
ν	=	0.3

The largest 19 coefficients were used to predict the buckling load and was found to be $\lambda = 0.6$. The experiment gave $\lambda = 0.554$.

4.0 BEHAVIOR OF KOITER'S PARAMETER

For most solution procedures used in the previous section, the imperfection sensitivity of a shell is based upon the evaluation of Koiter's parameter, b, in accordance with Eq. (73). In order to view this concept in proper perspective, it was decided to use program PVRCK and evaluate the minimum buckling load for each value of axial (m) and circumferential (n) Fourier harmonic. Since Donnell's equations are inaccurate for $n \leq 2$, no attempt should be made to extrapolate lower than, say, $n = 4$.

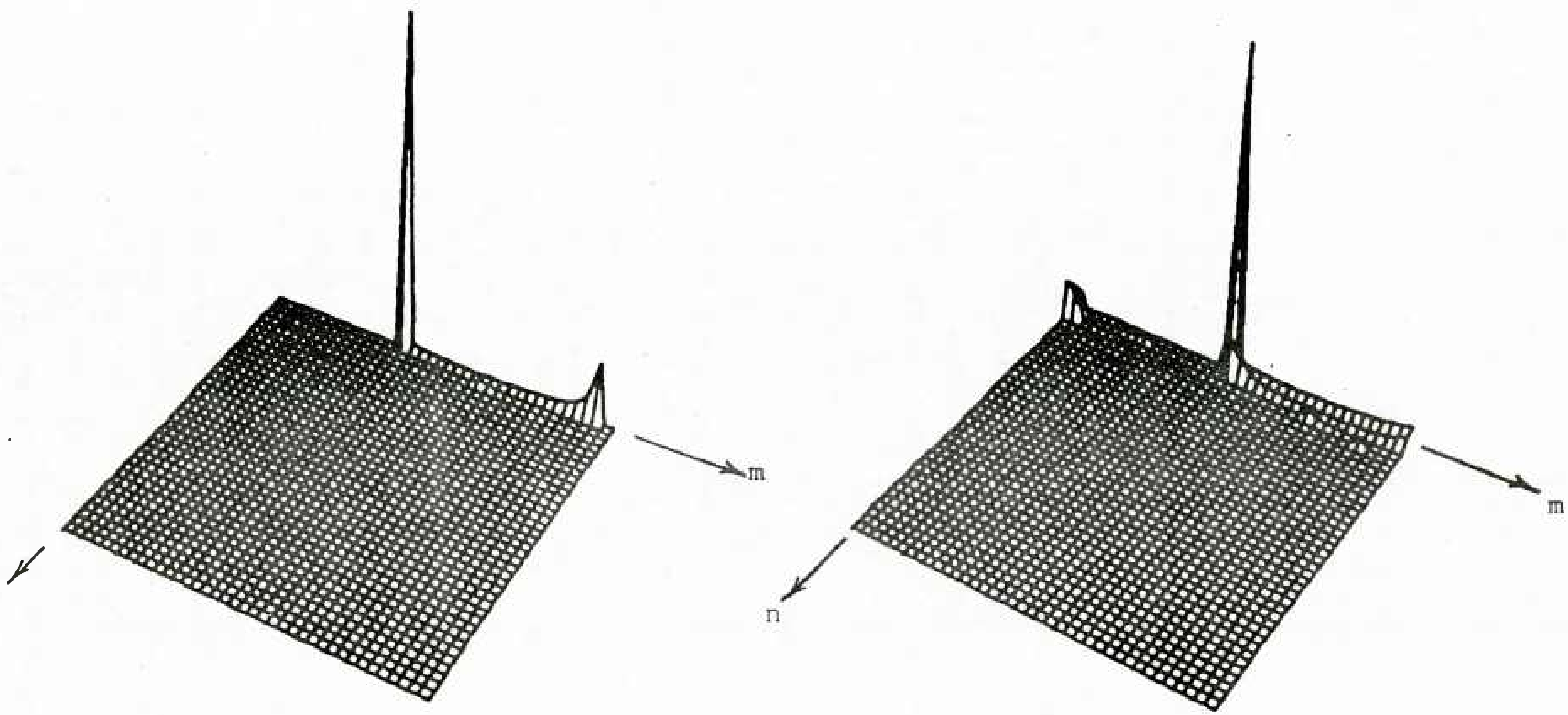
The selected solution procedure was to determine, for each value of m and n, the minimum buckling mode per Eq. (61), evaluate b per Eq. (73), and determine λ_s from Eq. (75). If b is positive, there is no reduction in the buckling load below the classical value. This procedure was followed for various ranges of Batdorf's parameter ($.1 \leq Z \leq 1000$) and imperfection geometries ($.0001 \leq \xi \leq 1.5$).

A three-dimensional perspective plot was generated for each minimum found. Figure 2 illustrates the radical behavior of b. For different values of Z, the behavior appears to be quite different. For $Z \leq 2$, the critical wave number of $n = 2$ was determined. Thus, the results for this region was ignored.

The trends of Koiter's parameter for various Batdorf's parameters show that at extremely low values of $Z(<2)$, b is either small (> -0.001) or positive. Thus the shell is insensitive to imperfections. Some moderate fluctuations occur $3 < Z < 10$. Note, in this same region Hutchinson predicted the greatest imperfection sensitivity; however, this was never confirmed by experiment.

For low values of Z one mode appeared to dominate. Above $Z > 20$, adjacent modes became just as dominant as the minimum mode. For higher wave numbers, the critical mode may not have the largest imperfection sensitivity factor.

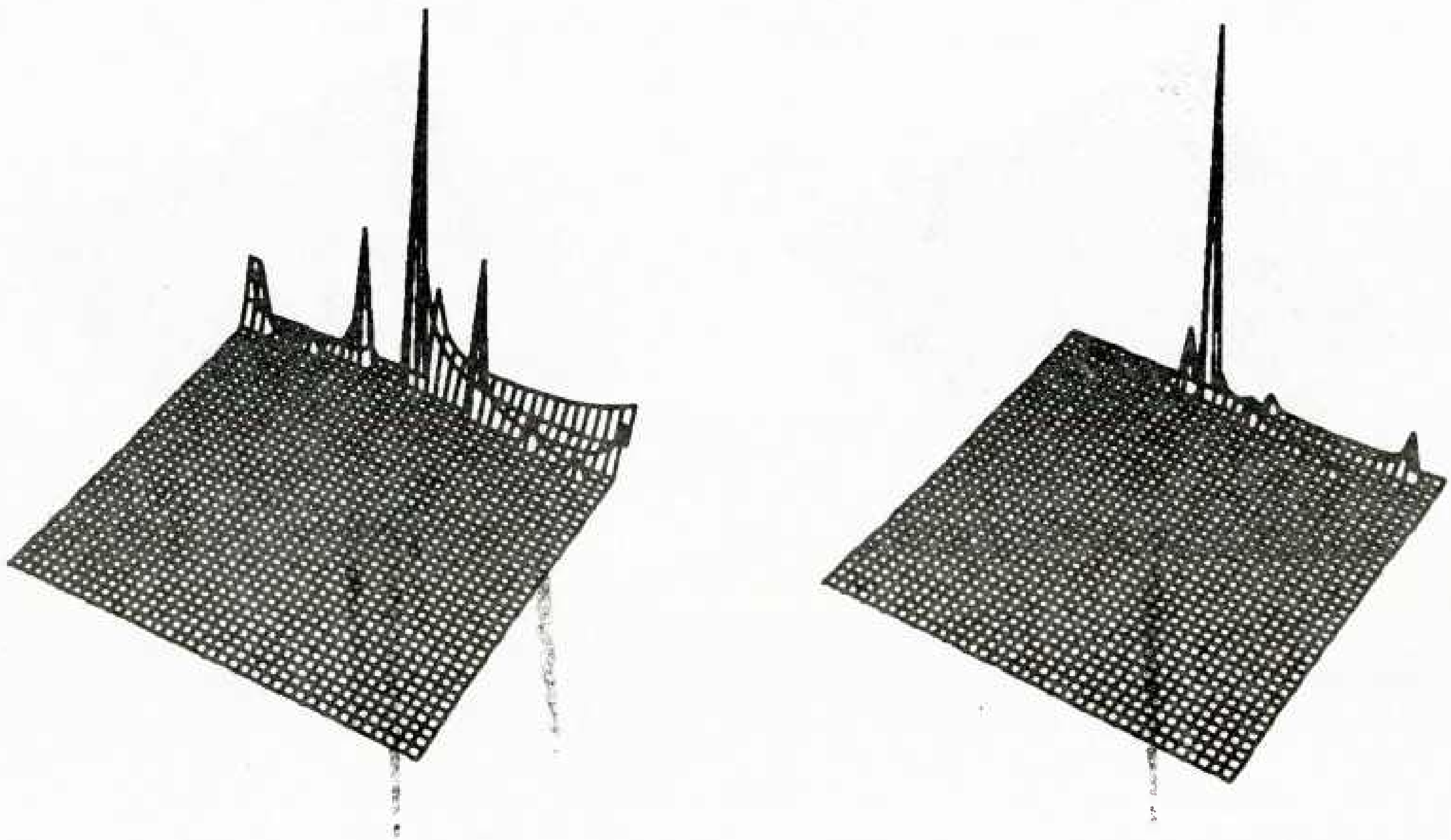
With increasing Z, the predicted critical axial mode number also increases, verifying Koiter's prediction [11]:



(a) Z = 5, b = - 2.29 x 10^{-4}

(b) Z = 7.5, b = - 1.46

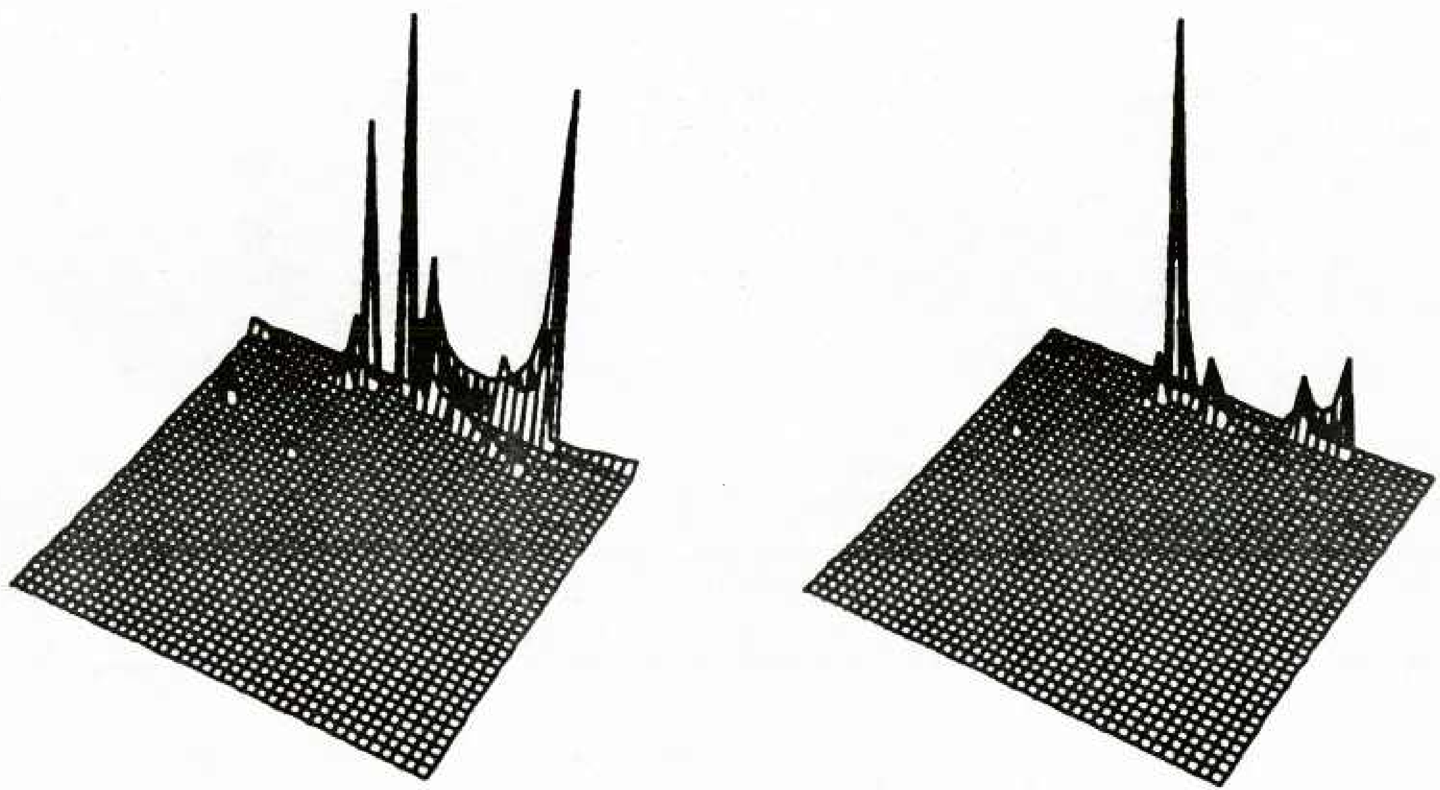
Figure 2 Koiter's Parameter for Axially Loaded Cylinder



(c) Z = 10.0, b = -1.237

(d) Z = 15, b = - 2.51

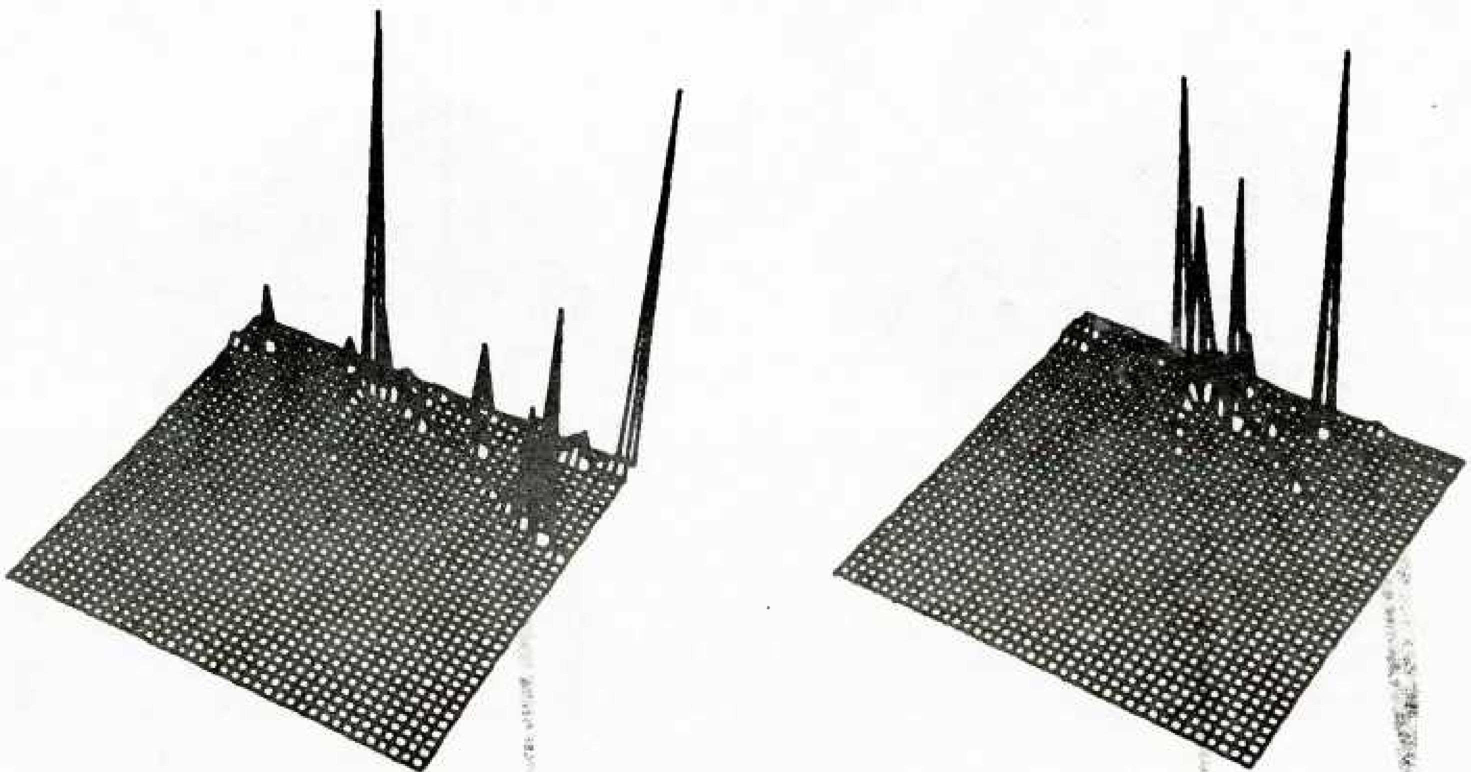
Figure 2 (continued)



(e) Z = 20, b = - 4.2

(f) Z = 30, b = - 10.98

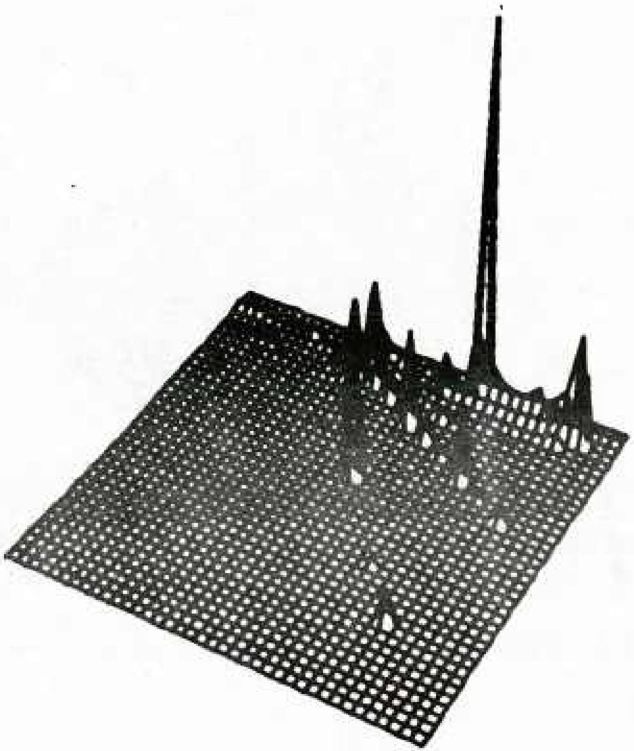
Figure 2 (continued)



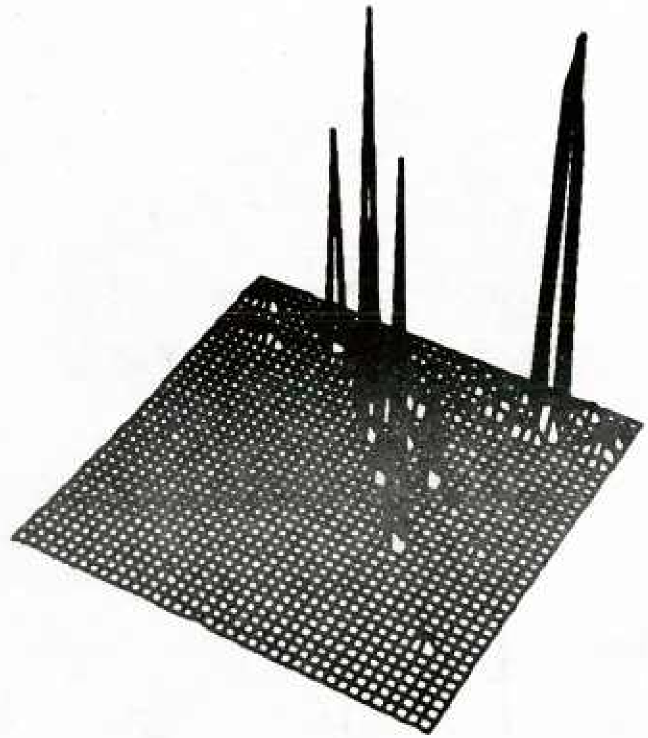
(g) Z = 50, b = - 20.09

(h) Z = 75, b = - 2.05

Figure 2 (continued)

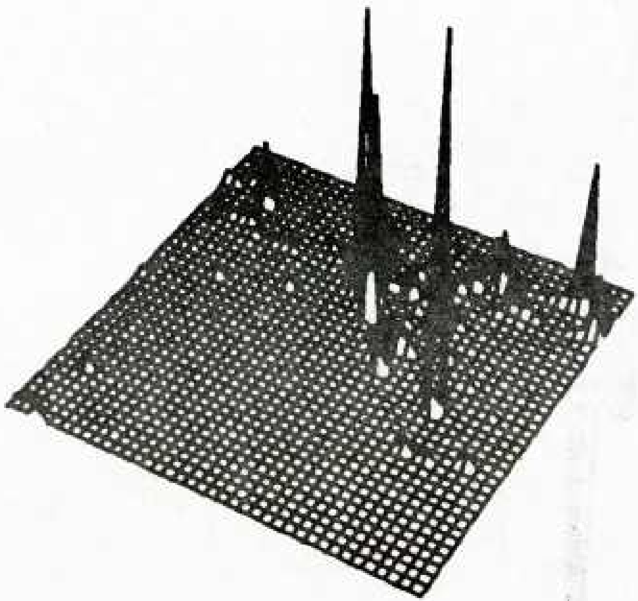


(i) $Z = 100$, $b = -25.61$

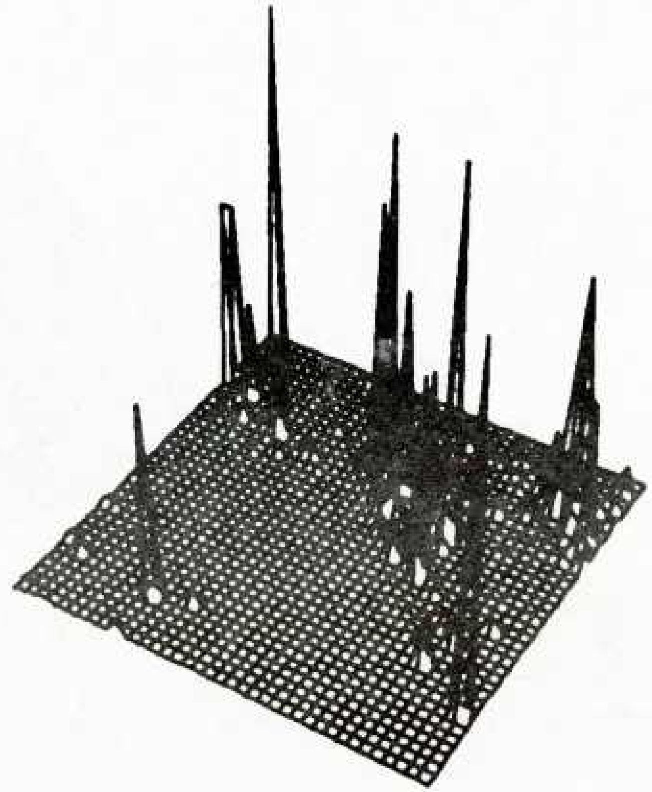


(j) $Z = 250$, $b = -4.82$

Figure 2 (continued)

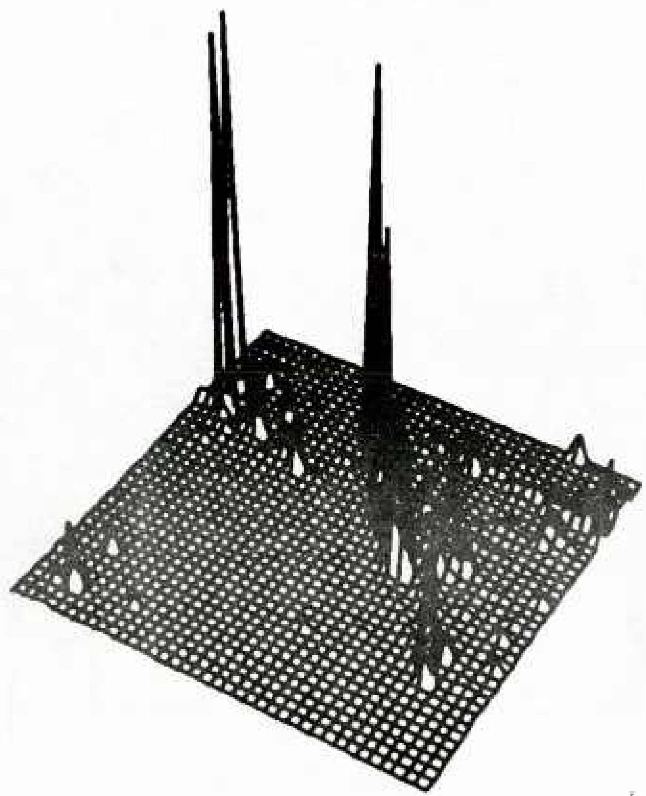


(k) Z = 500, b = - 20.27



(ℓ) Z = 750, b = - 6.28

Figure 2 (continued)



(m) Z = 1000, b = - 10.66

Figure 2 (continued)

$$m_{cr} = \frac{2\sqrt{3}}{\pi^2} \sqrt{Z}.$$

Since these results are based upon the assumption that boundary conditions have little influence, it is anticipated that the predictions are more conservative than those obtained with the proper boundary conditions. However, the magnitude of Koiter's parameter which corresponds to the critical mode (denoted by Δ in Fig. 3) has a very large magnitude and is quite erratic. If the near minimum buckling load is chosen on the basis of the minimum ordered mode (denoted by o in Fig. 3 which is at the least value of the axial wave number), then the predictions of b become quite well behaved.

This same observation of erratic behavior of Koiter's parameter was reported by Yamaki [19]. Two different analysis procedures were used (an asymptotic and a full nonlinear method) and the results obtained were similar to those presented herein. Yamaki's observation was that the lower bound of the critical loads diminishes significantly with the increase of Z, as well as R/t. For long shells with $Z > 500$, the imperfection sensitivity factor remains constant. Further, the postbuckling mode is always symmetric for short shells $Z < 100$; while for long shells with $Z > 200$ asymmetric modes are predominant. The wave number immediately after buckling is generally smaller than the critical wave number. The imperfection sensitivity parameter for pure torsion and hydrostatic pressure as a function of Z was well behaved, while for compression some erratic behavior was observed.

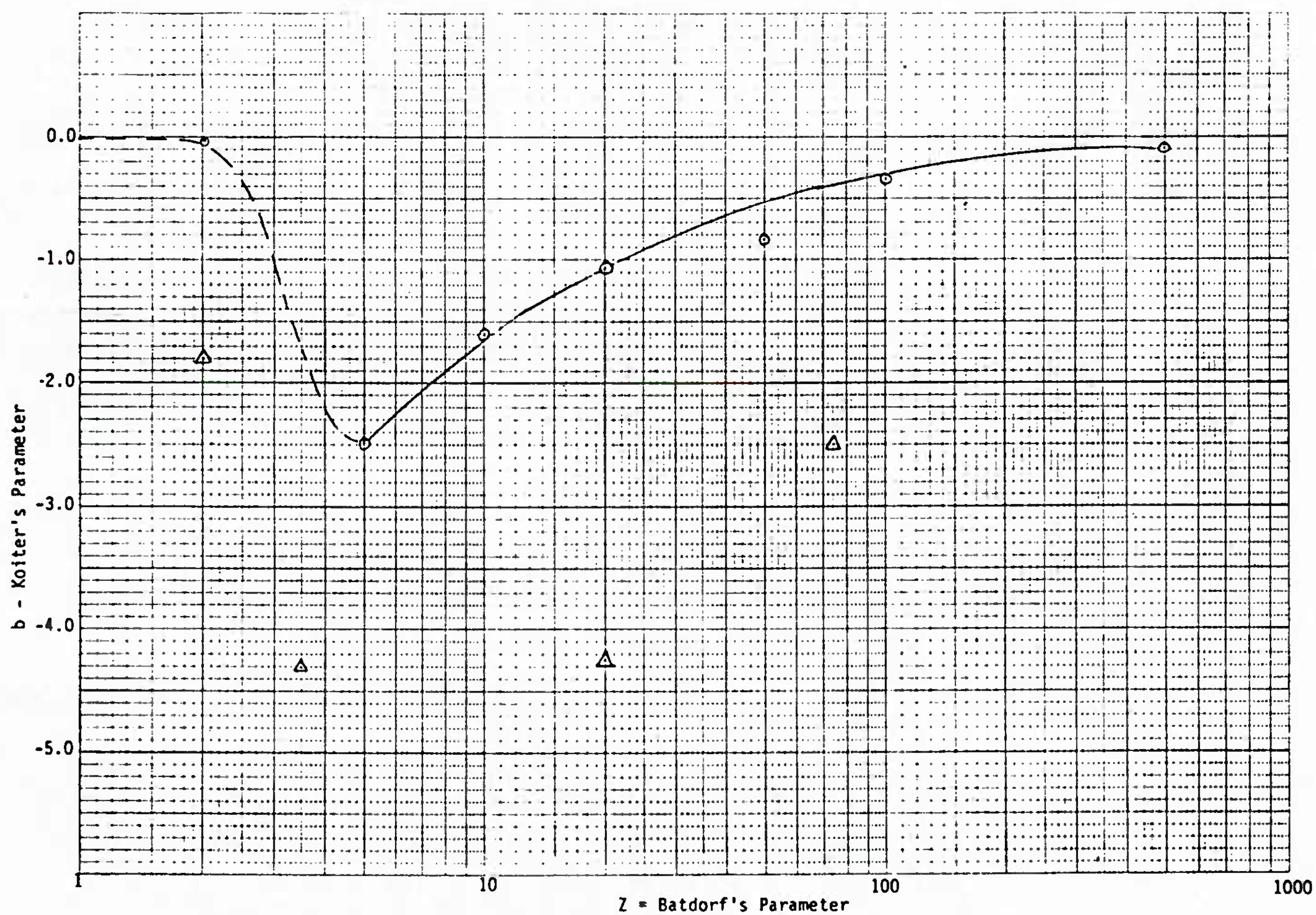


Figure 3   Variation of Koiter's Parameter for Axially Loaded Cylinder

5.0 CONCLUSIONS AND OBSERVATIONS

Four computer program procedures have been assembled that permit a convenient and economical evaluation of the imperfection sensitivity of cylindrical shells. When all of the imperfection data is known, reasonable agreement to experimental results is found for these programs. Because of the basic assumptions behind each program, certain caution must be taken in attempting to apply the results directly. Similarly, extreme care should be taken when evaluating any experimental data because not all of the significant data may be reported.

During the assembly and verification of the computer programs, a variety of test cases were run. From these solutions, some observations on the factors that affect behavior of the imperfection sensitivity of shells were made. Some of these observations are noted as follows:

(1) For the general types of membrane producing loads (axial compression, torsion, and hydrostatic pressure), axial compression is associated with the greatest imperfection sensitivity.

(2) When torsion is combined with axial compression, even greater imperfection sensitivity is observed.

(3) The effects of boundary conditions may mask imperfection sensitivity, particularly for axially stiffened shells.

(4) Shells with outside ring stiffeners tend to develop more imperfection sensitivity than those with internal ring stiffeners.

(5) Two-way stiffening is far more effective than single direction stiffening.

(6) For certain cases, a multiplicity of eigenvalues can exist at the same buckling load. For those states of stress, the imperfection sensitivity corresponding to these adjacent eigenvalues can be considerably different.

(7) No logical conclusions can be made for one shell geometry which are applicable to all types of cylindrical sizes and loading conditions. More extensive parametric studies should be performed in order to draw general conclusions.

(8) Far more detailed experimental evidence is required to fully demonstrate a particular behavior of imperfection sensitivity. The effects of axisymmetric imperfection have been fairly adequately documented. The effects of asymmetric imperfection have not.

The degradation of the critical axial compression load not only depends upon the value of the imperfection sensitivity factor but also on the magnitude of the imperfection and its waveform. Single mode behavior, particularly for the asymmetric mode, suggests that only small imperfection amplitudes ought to be considered when employing Koiter's method. Further multi-mode participation will have a much greater affect on the critical buckling mode than single imperfection geometry.

The study of imperfection sensitivity of shells cannot be adequately resolved by simply modifying classical linear buckling theories or even nonlinear buckling theories. A more unified approach of handling imperfection geometry is required. When a complete description of an imperfection geometry is established, adequate correlation to experimental test results can be obtained. There is no current accepted method that can be used to suggest, in advance, which critical participating modes should be used in the analysis. At present, this can be accomplished only through trial and error.

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